Advances and perspectives in numerical modelling using Serre-Green Naghdi equations

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$$\mu = \left(\frac{d_0}{\lambda_0}\right)^2$$ small

$$\varepsilon = \frac{A_0}{d_0} = O(1)$$

Serre or **Green Naghdi** equations (**S-GN**):

basic weakly dispersive fully nonlinear **Boussinesq** equations (order O(μ))

OUTLINE

- Introduction
- History of S-GN equations
- Ability and limitations of classical FD approaches for S-GN
- A new approach for solving S-GN
  - reformulation of S-GN equations
  - hybrid FV/FD methods
  - shock wave modelling for wave breaking
- Results
- Conclusions

Acknowledgments

❑ Fluid mechanics and nearshore dynamics

- Eric Barthélémy (LEGI, Grenoble)
- Rodrigo Cienfuegos (PUC, Santiago de Chile)
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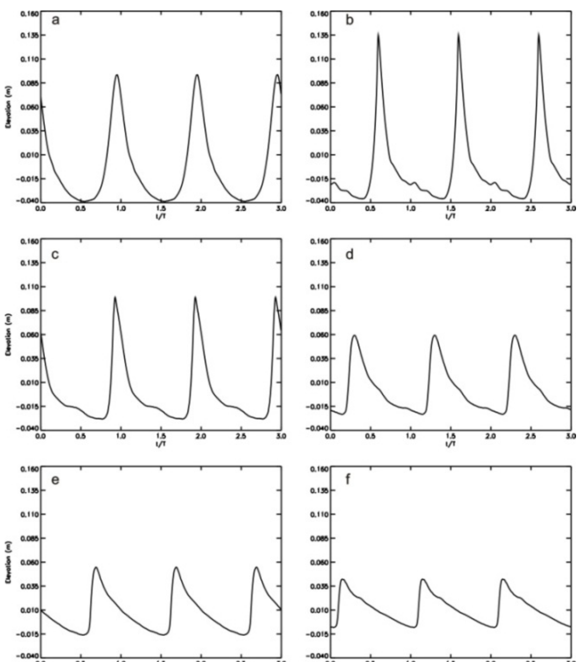
❑ Mathematics

- David Lannes (ENS, Paris) → nonlinear PDE theory
- Fabien Marche (I3M, Montpellier) → numerical methods
- Florent Chazel (INSA, Toulouse) → numerical methods

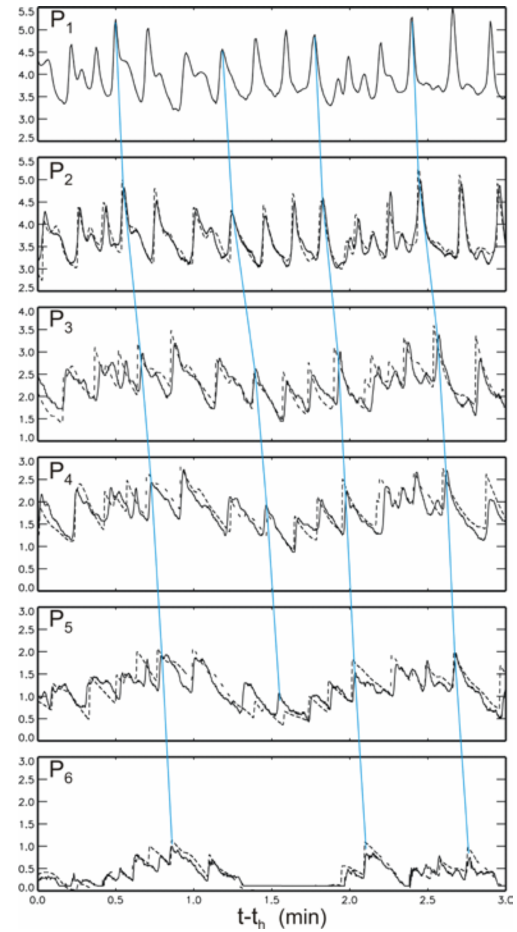
Introduction

Long wave propagation in the nearshore is most often associated with strongly **nonlinear** processes

- shoaling and breaking



- swash motions



Introduction

Long wave propagation in the nearshore is most often associated with strongly **nonlinear** processes

- tsunamis

Sumatra 2004 tsunami reaching the coast of Thailand (from Madsen et al.2008)

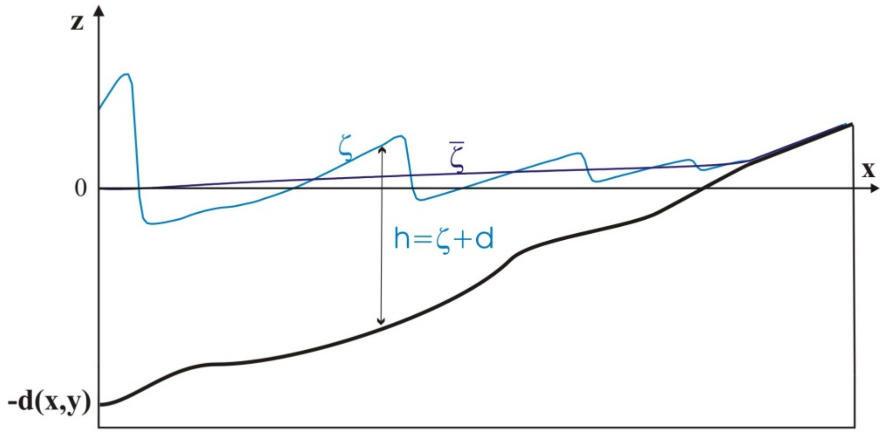
Fully nonlinear weakly dispersive approaches are required

⇨ **Serre – Green Naghdi** equations

History of the Serre – Green Naghdi equations

$$\partial_t \zeta + \nabla \cdot (h\mathbf{u}) = 0$$

$$\partial_t \mathbf{u} + \varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\zeta = \mu\mathcal{D} + O(\mu^2)$$



ε=O(1)

1D equations, flat bottom

- **Serre** (1953)

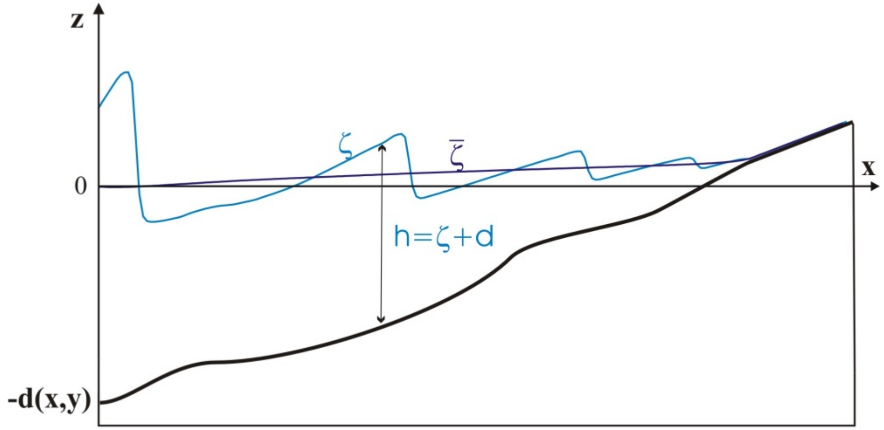
$$\mathcal{D} = \frac{1}{3h}\partial_x(h^3(u_{xt} + \varepsilon u u_{xx} - \varepsilon(u_x)^2))$$

→ closed form solutions for fully nonlinear solitary and cnoidal waves
(see Barthémémy, 2004 and Carter and Cienfuegos, 2011)

- Su and Garner (1969), Venezian (1974)

History of the Serre – Green Naghdi equations

$$\begin{aligned} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) &= 0 \\ \partial_t \mathbf{u} + \varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\zeta &= \mu\mathcal{D} + O(\mu^2) \end{aligned}$$



$\varepsilon = O(1)$

1D equations, uneven bottom

- Seabra-Santos et al. (1987)
- Guibourg and Barthélémy (1994)

Dingemans controversy

“ vertical uniformity assumption for v “ ?

MAST-G8M, 1994 and book, part 2,1997

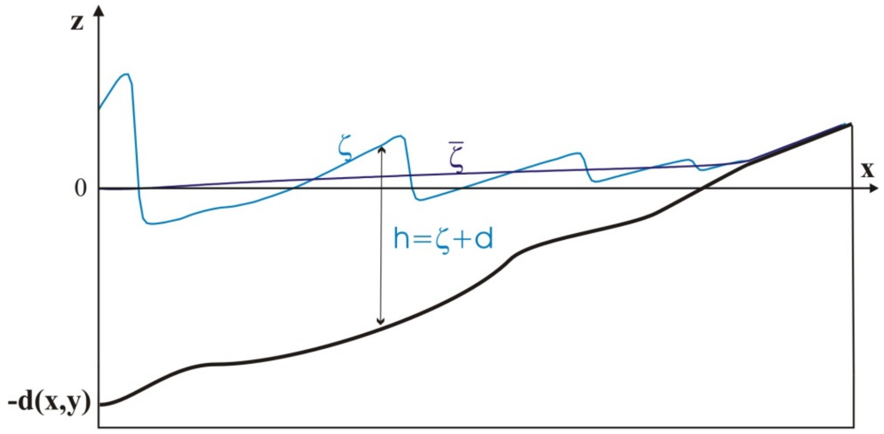
$$\int_{-d}^{\varepsilon\zeta} \left(v^2(x,z) - u^2\right) dz = O(\mu^2)$$

Cienfuegos et al., 2006

⇨ validity of Serre equations for uneven bottom

History of the Serre – Green Naghdi equations

$$\begin{aligned} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) &= 0 \\ \partial_t \mathbf{u} + \varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\zeta &= \mu\mathcal{D} + O(\mu^2) \end{aligned}$$



$\varepsilon = O(1)$

2D equations, uneven bottom

- **Green and Naghdi** (1976)
- Mei (1983), horizontal bottom
- Wei et al. (1995): improved dispersive properties

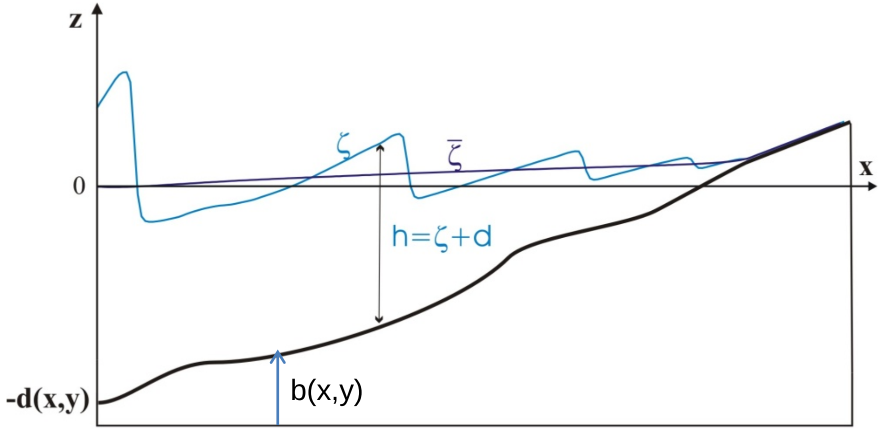
  $\mathbf{u}_\alpha(x,t)=\mathbf{v}(\mathbf{x},t,z_\alpha(\mathbf{x},t))$ (*Nwogu, 1993)* → open source FUNWAVE code
- Many other studies on S-GN in Physics and Mathematics:

  e.g.: Carter, Dias, Dutykh, El, Gavrilyuk, Grimshaw, Israwi, Lannes, Li, Miles, Smyth, …

Serre – Green Naghdi equations

$$\begin{aligned} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) &= 0 \\ \partial_t \mathbf{u} + \varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\zeta &= \mu\mathcal{D} + O(\mu^2) \end{aligned}$$

Lannes and Bonneton (2009)



$$\mathcal{D} = -\mathcal{T}[h,b]\mathbf{u}_t - \varepsilon\mathcal{Q}[h,b](\mathbf{u})$$

where the linear operator $\mathcal{T}[h,b]$ is defined as

$$\mathcal{T}[h,b]W = -\frac{1}{3h}\nabla(h^3\nabla\cdot W) + \frac{1}{2h}[\nabla(h^2\nabla b\cdot W) - h^2\nabla b\nabla\cdot W] + \nabla b\nabla b\cdot W$$

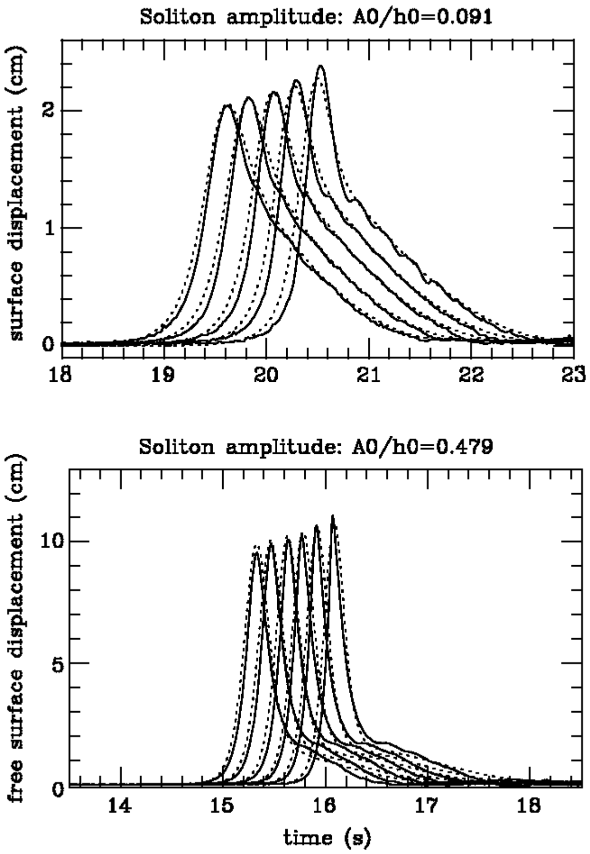
and the quadratic term $\mathcal{Q}[h,b](\mathbf{u})$ is given by

$$\begin{aligned} \mathcal{Q}[h,b](\mathbf{u}) &= -\frac{1}{3h}\nabla\left(h^3((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u}) - (\nabla\cdot\mathbf{u})^2)\right) \\ &+\frac{1}{2h}[\nabla(h^2(\mathbf{u}\cdot\nabla)^2 b) - h^2((\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u}) - (\nabla\cdot\mathbf{u})^2)\nabla b] + ((\mathbf{u}\cdot\nabla)^2 b)\nabla b \end{aligned}$$

Ability and limitation of classical FD approaches for S-GN

Seabra-Santos et al.(1987) and Guibourg and Barthélémy (1994)

- implicit FD scheme (*Su and Mirie, 1980*)



ε_0~0.1

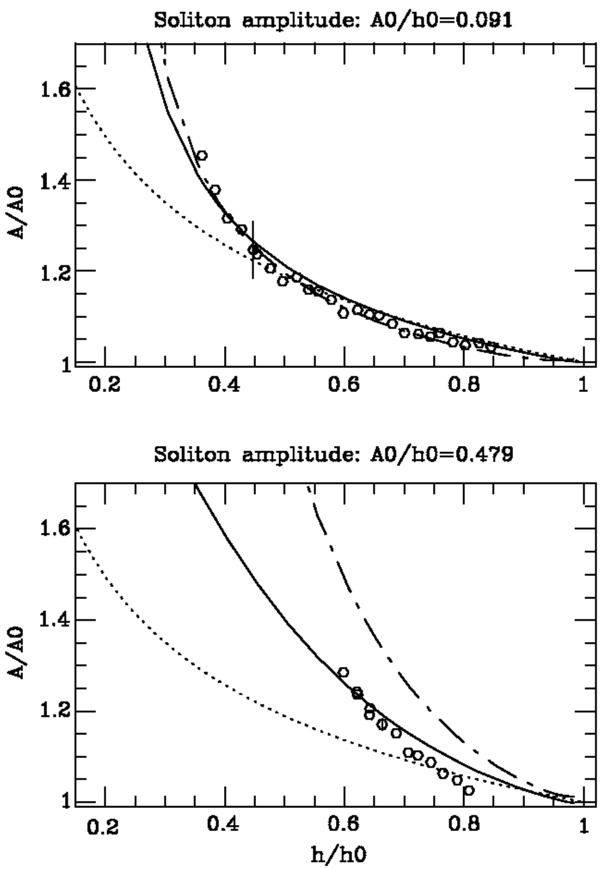
ε_0~0.5

Figure 9. Free surface displacements against time (in seconds) for a shoaling solitary wave at various locations. Beach slope of 1/60. (—) flume measurements; (- -) numerical simulations by the Serre equations. A_0 is the amplitude of the solitary wave at the toe of the beach and the h_0 the uniform depth before the beach.

Barthélémy (2004)

Seabra-Santos et al.(1987) and Guibourg and Barthélémy (1994)

- implicit FD scheme (*Su and Mirie, 1980*)



$\varepsilon_0 \sim 0.1$

$\varepsilon_0 \sim 0.5$

Figure 10. Wave peak amplitude A evolution along the beach. h is the depth at a given position on the beach. (o) Flume measurements; (–) numerical simulations by the Serre equations; (-·) numerical simulations by the Boussinesq equations; (· · ·) Green's law, equation (60).

Barthélémy (2004)

Wei et al. (1995) and Kirby et al. (1998) → **FUNWAVE code**

- high-order FD scheme: 4th order in time and mixed-order 2nd and 4th order in space

ε_0=0.2

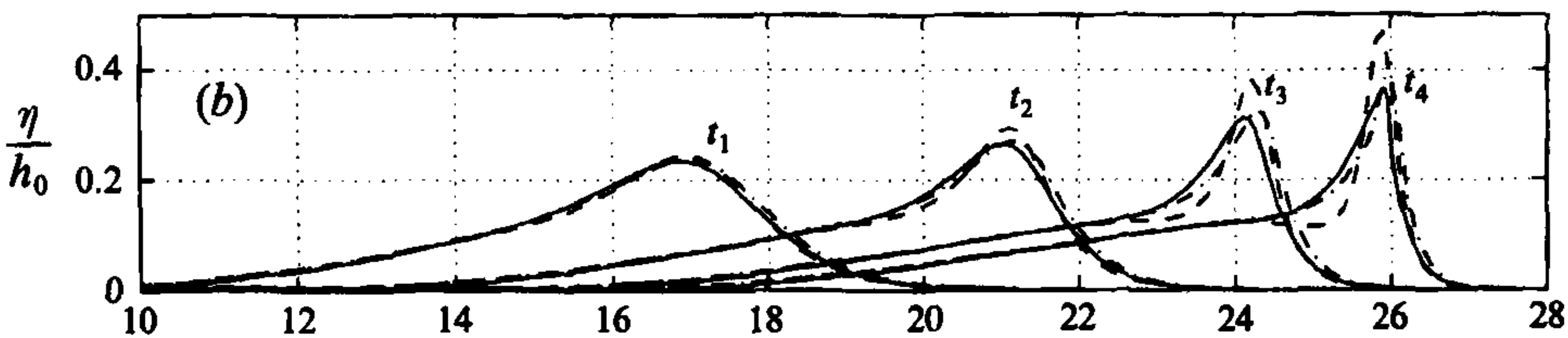


FIGURE 4. Comparison between FNPF (——), BM (- - - - -), and FNBM (— - —) of free surface elevations for the shoaling of solitary waves, with $\delta = 0.2$ in (*a*), (*b*), and (*d*) and 0.3 in (*c*). (*a*) $s = 1{:}100$; $t' = t_1 = 39.982$, $t_2 = 53.191$, $t_3 = 61.131$, $t_4 = 66.890$; (*b*) $s = 1{:}35$; $t' = t_1 = 16.243$, $t_2 = 20.640$, $t_3 = 24.032$, $t_4 = 25.936$; (*c*) $s = 1{:}15$; $t' = t_1 = 3.230, t_2 = 6.000$, $t_3 = 8.401$, $t_4 = 11.320$; (*d*) $s = 1{:}8$; $t' = t_1 = -0.739$, $t_2 = 2.575$, $t_3 = 5.576 = t_4 = 6.833$. The last FNPF profile in (*a*)–(*c*) corresponds to the theoretical breaking point for which the wave front face has a vertical tangent.

Wei et al. (1995) and Kirby et al. (1998) → **FUNWAVE code**

- high-order FD scheme: 4th order in time and mixed-order 2nd and 4th order in space

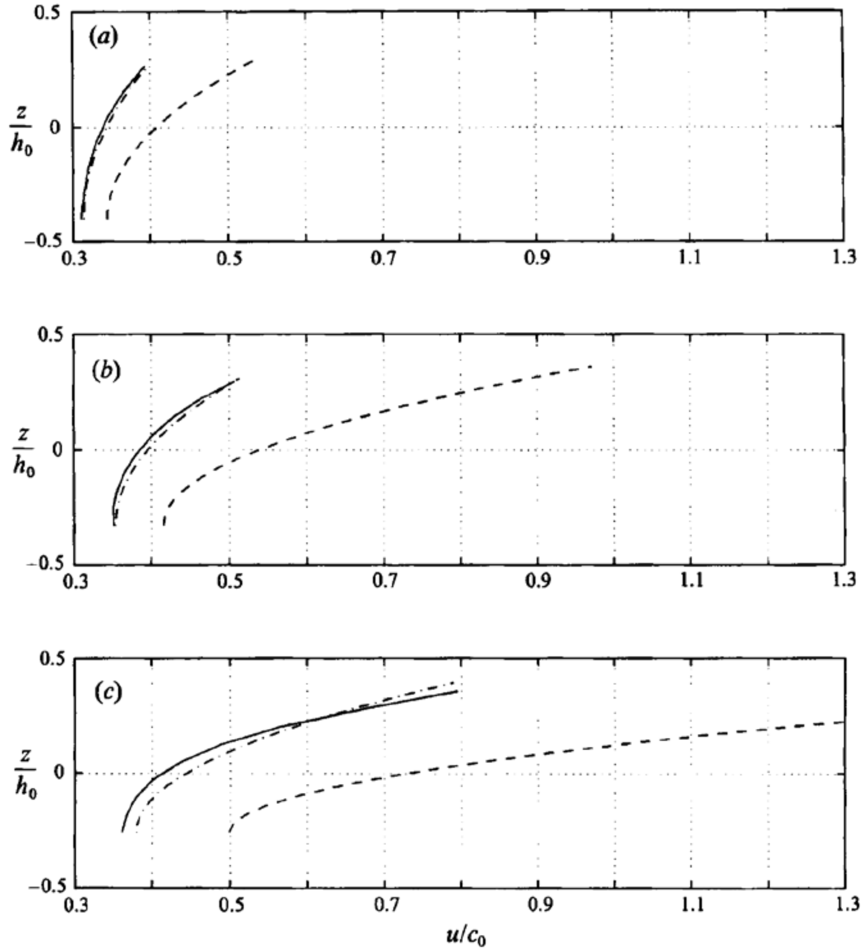


FIGURE 8. Comparison between FNPF (——), BM (- - - - -), and FNBM (— - —) of horizontal velocity profiles with initial height $\delta = 0.2$ on slope 1:35 at different locations: (*a*) $x' = 20.96$; (*b*) $x' = 23.63$; (*c*) $x' = 25.91$.

- **Zelt (1991)**: eddy viscosity analogy → ad hoc extra term in the momentum equation

- Application to S-GN equations: **Kennedy et al. (2000)**

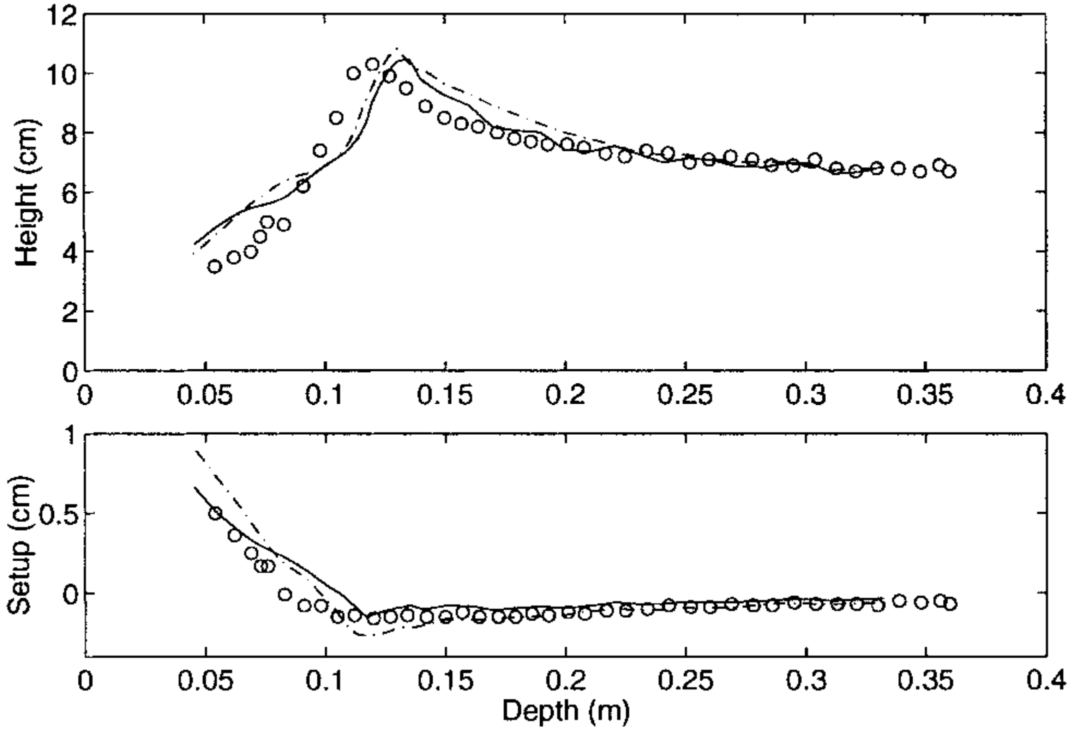
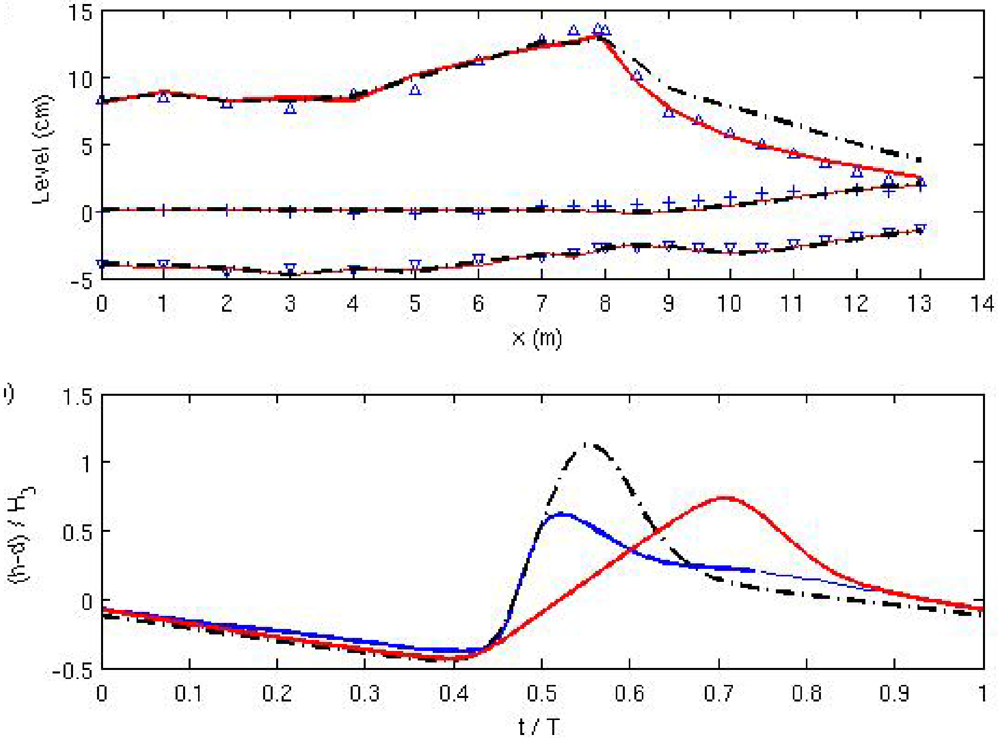


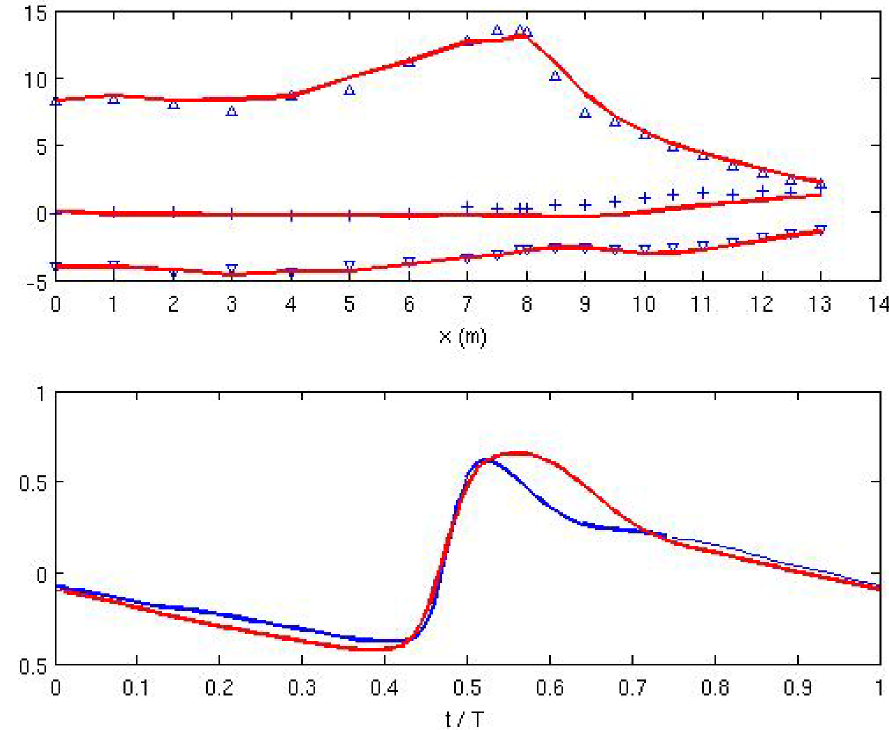
FIG. 5. Computed and Measured Wave Heights and Setup for Hansen and Svendsen Spilling Breaker 061071: Data (○); WKGS (——); Nwogu (–·–)

Cienfuegos et al. (2010)

- Kennedy et al. (2000) parametrization
- eddy viscous diffusion terms on both, the momentum and mass equations

Validation with Ting and Kirby (1994) spilling breaking experiments





⇨ wave asymmetry is lost

⇨ wave asymmetry is correctly reproduced

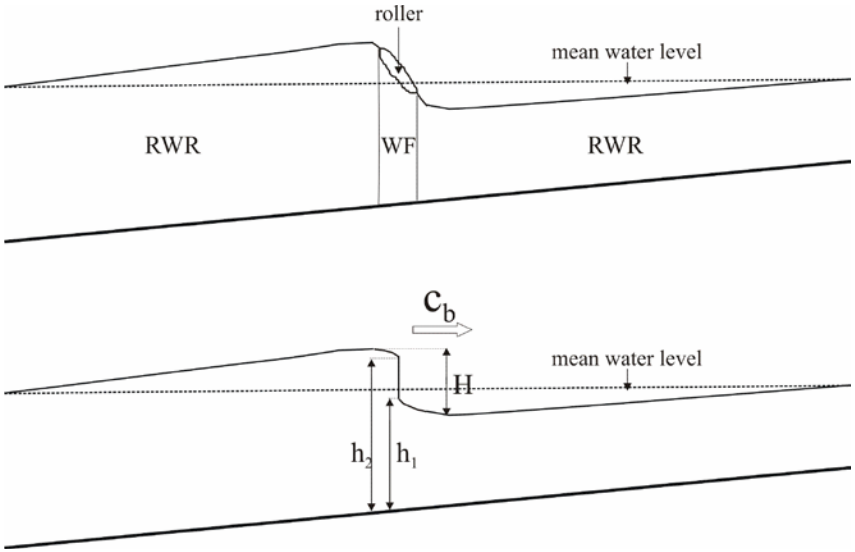
a lack of well-posed theoretical and numerical approaches to handle wave breaking dissipation and shoreline motions

- numerical filtering is required to avoid noisy results or numerical instabilities (Shi et al., 2012)
- tuning of several parameters as the ones determining wave-breaking dissipation and run-up (Bruno et al., 2009)
- propagation over complex bathymetries, swash motions and wave overtopping

NSWE and shock waves

Nonlinear shallow water equations (NSWE) give a good theoretical framework for broken-wave energy dissipation and shoreline motions

Wave fronts are represented by shock waves

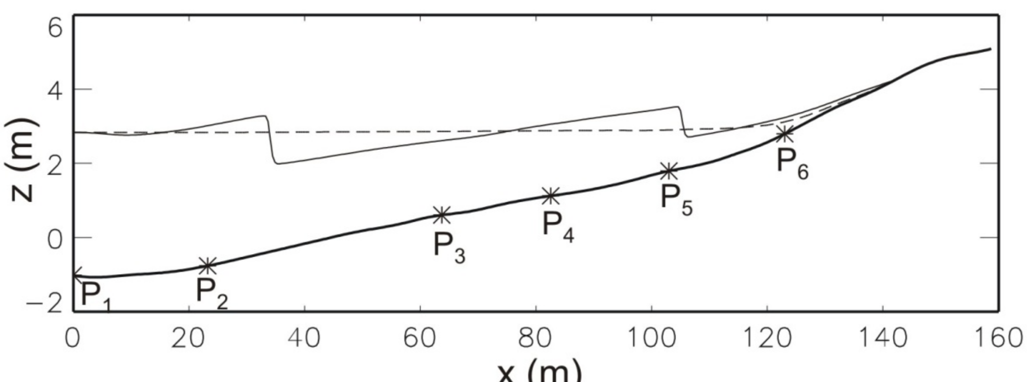


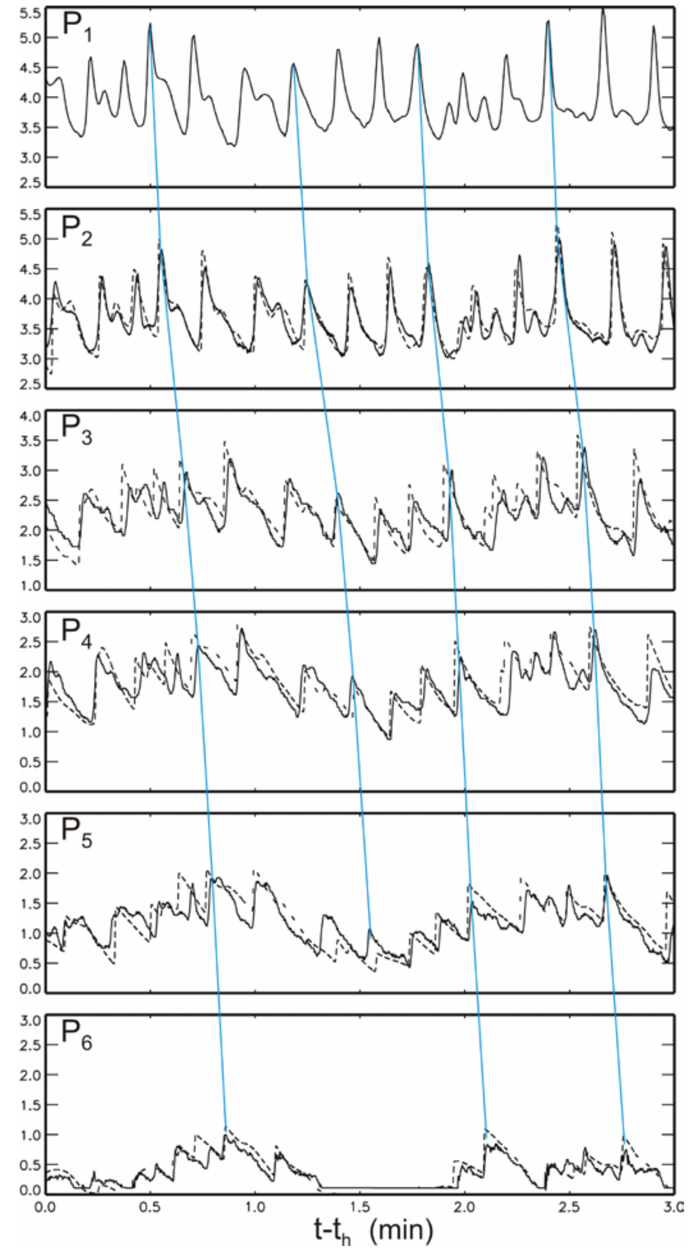
Energy dissipation is given by mass and momentum conservation across the shock

NSWE and shock waves

NSWE and shock wave approach give good results for: energy dissipation, broken-wave celerity and swash motion

see Hibbert and Peregrine (1979), Kobayashi et al. (1989), Raubenheimer et al. (1996), many others, …





Bonneton et al (2004)

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{q})}{\partial x} = \mathbf{S}(\mathbf{q})$$

$$\mathbf{q} = \begin{pmatrix} d \\ h \\ hu \end{pmatrix}, \mathbf{F} = \begin{pmatrix} 0 \\ hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 0 \\ 0 \\ gh\frac{\partial d}{\partial x} \end{pmatrix}$$

Recent advances in NSWE modelling

see Marcel Zijlema's talk

- accurate wave front simulations
  - high-order finite volume shock-capturing methods
  - e.g. Leveque (2002), Bouchut (2004) or Toro (2009)
- wetting and drying processes
  - water depth positivity preserving schemes
  - e.g. Gallouet et al. (2003), Berthon and Marche (2008)
- strongly varying bathymetries
  - Greenberg and Le Roux (1996): exact Riemann solvers on varying topography
  - well-balanced schemes
  - e.g. Zhou et al. (2001) or Audusse et al. (2004)
- no need of numerical filtering

A new approach for modelling nonlinear waves in the nearshore

$$\begin{aligned} \partial_t \zeta + \nabla \cdot (h\mathbf{u}) &= 0 \\ \partial_t \mathbf{u} + \varepsilon(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla \zeta &= \cancel{\mu \mathcal{D}} \end{aligned}$$

Fully nonlinear

non-breaking waves:

S-GN equations

Broken wave fronts

and swash motions:

NSWE

A new approach for solving S-GN

An hybrid finite volume / finite difference scheme

$$
\begin{aligned}
\partial_t \zeta + \nabla \cdot (h\mathbf{u}) &= 0 \\
\partial_t \mathbf{u} + \varepsilon(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla\zeta &= \mu\mathcal{D}
\end{aligned}
$$

FV shock-capturing scheme FD scheme

- weakly nonlinear Boussinesq equations:

Soares-Frazao and Zech (2002), Erduran et al. (2007), Tonelli and Petti (2009), Orszaghova et al. (2012),…

- S-GN equations:

$\rightarrow$ SURF-GN code : *Tissier et al. (ICCE 2010), Chazel, Marche and Lannes (2011), Bonneton et al. (JCP and EJM/B, 2011), Tissier et al. (2012)*

$\rightarrow$ FUNWAVE-TVD code: *Shi et al. (2012)*

$$\partial_t h + \nabla\cdot(h\mathbf{u}) = 0$$

$$\partial_t(h\mathbf{u}) + \nabla\cdot(h\mathbf{u}\otimes\mathbf{u}) + \nabla(\frac{1}{2}gh^2) = -gh\nabla b$$

$$+\frac{1}{\alpha}gh\nabla\zeta - (I + \alpha h\mathcal{T}\frac{1}{h})^{-1}[\frac{1}{\alpha}gh\nabla\zeta + h\mathcal{Q}_1(\mathbf{u})]$$

$\mathcal{Q}_1(\mathbf{u}) = \mathcal{Q}(\mathbf{u}) - \mathcal{T}((\mathbf{u}\cdot\nabla)\mathbf{u})$ only involves second order derivatives of $\mathbf{u}$

α −− > improved dispersive properties (Madsen et al., 1991)

$kd_0 \leq 3$

*Bonneton, Chazel, **Lannes**, Marche and Tissier (2011)*

SURF-GN Code

$$S(\Delta t)=S_1(\Delta t/2)\, S_2(\Delta t)\, S_1(\Delta t/2)$$

$S_1(t)$: hyperbolic part of the S-GN equation

$$\begin{aligned} \partial_t h + \nabla\cdot(h\mathbf{u}) &= 0 \\ \partial_t(h\mathbf{u}) + \nabla\cdot(h\mathbf{u}\otimes\mathbf{u}) + \nabla(\frac{1}{2}gh^2) &= -gh\nabla b \end{aligned}$$

- SURF-WB code *(**Marche** et al., 2007 ; Berthon and Marche, 2008)*
- positive preserving VFRoe scheme with 4^{th} order MUSCL reconstruction
- well-balanced scheme, hydrostatic reconstruction *(Audusse et al, 2004)*
- 4^{th} order Runge-Kutta scheme

$S_2(t)$: dispersive part of the S-GN equation

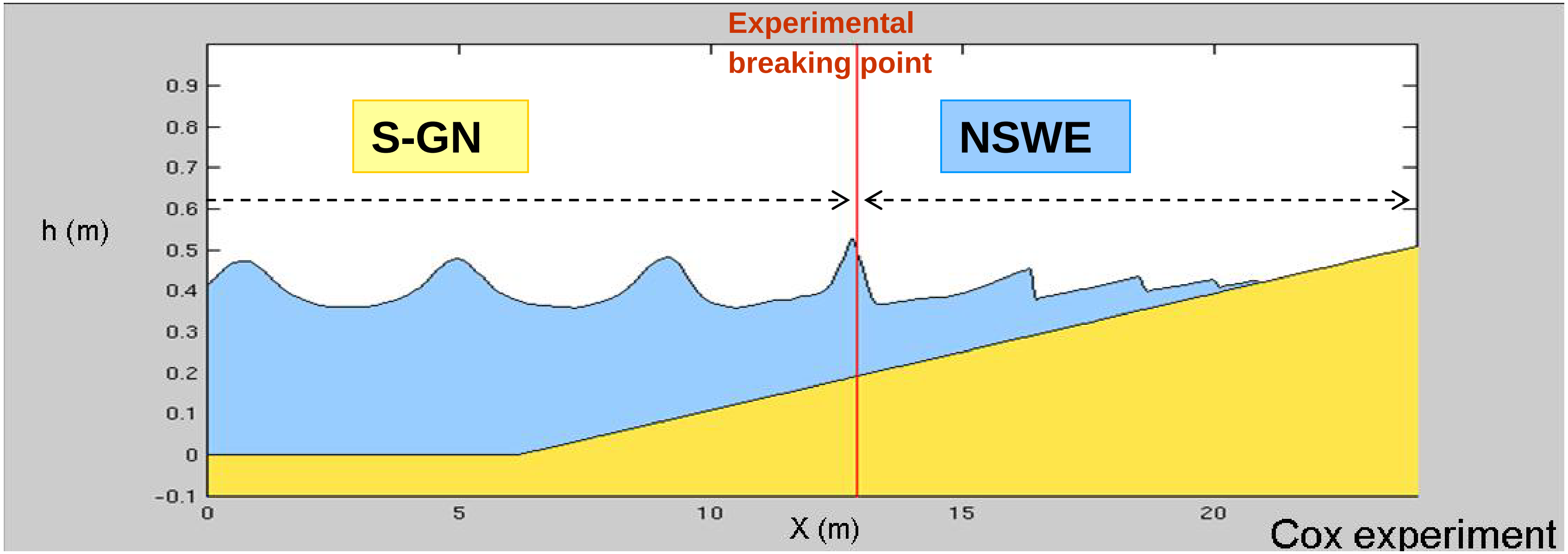
$$\begin{aligned} \partial_t h &= 0 \\ \partial_t(h\mathbf{u}) &= \frac{1}{\alpha}gh\nabla\zeta - (I+\alpha h\mathcal{T}\frac{1}{h})^{-1}[\frac{1}{\alpha}gh\nabla\zeta + h\mathcal{Q}_1(\mathbf{u})] \end{aligned}$$

- 4^{th} order in space and time

Strategy for wave breaking: description of broken-wave fronts as shocks by the NSWE, by skipping the dispersive step S2

Spatial decomposition

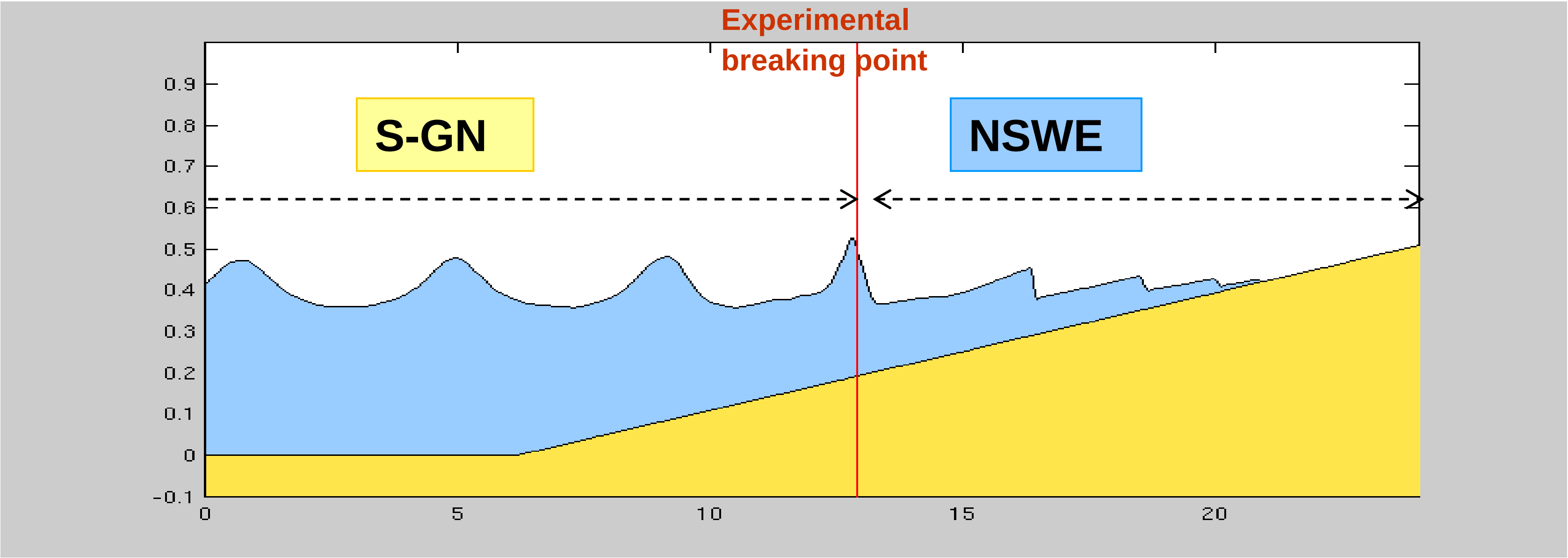
- *WN Boussinesq: Tonelli and Petti (2009), Orszaghova et al. (2012)*
- *S-GN: Tissier et al. (2010)*



no parametrization of wave breaking

→ only momentum and mass conservation across the shock

Spatial decomposition



no parametrization of wave breaking

→ only momentum and mass conservation across the shock

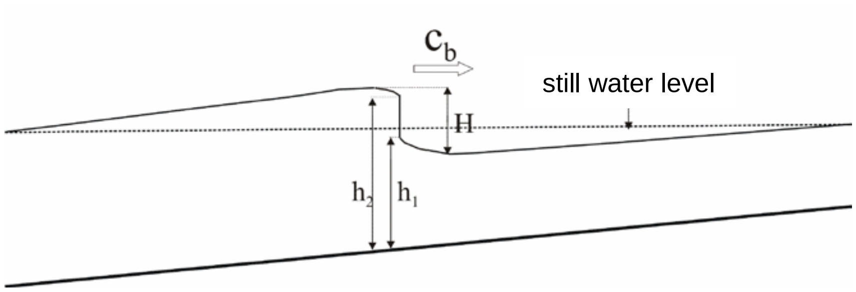
breaking termination (bar/trough) or Irregular waves → different breaking points

⇨ **local treatment of breaking**

❑ Tonelli and Petti (2009, 2010, 2011, 2012)

wealky nonlinear Boussinesq equations

$$\gamma = \frac{H}{d} > 0.8$$



The value of γ is computed and checked in each cell of the domain at every time step: if γ exceeds 0.8, the solution locally and temporary shifts from Boussinesq to NSWE

❑ Shi et al. (2012) → FUNWAVE-TVD code

S-GN equations

$$\gamma_L = \frac{\zeta}{d} > 0.8$$

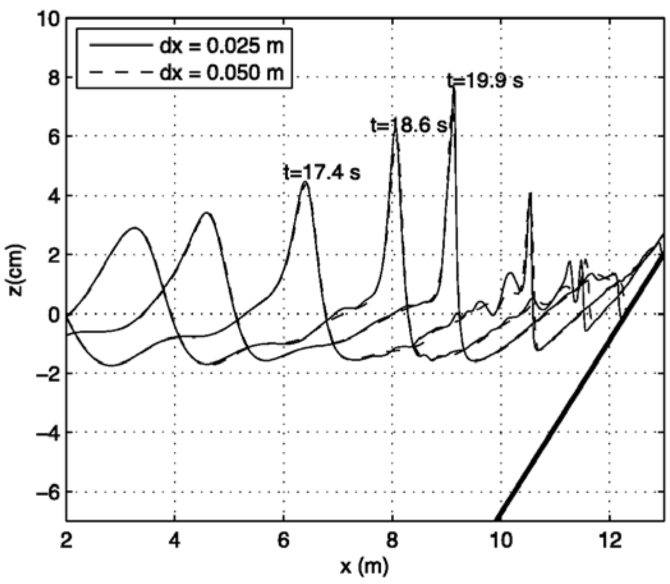
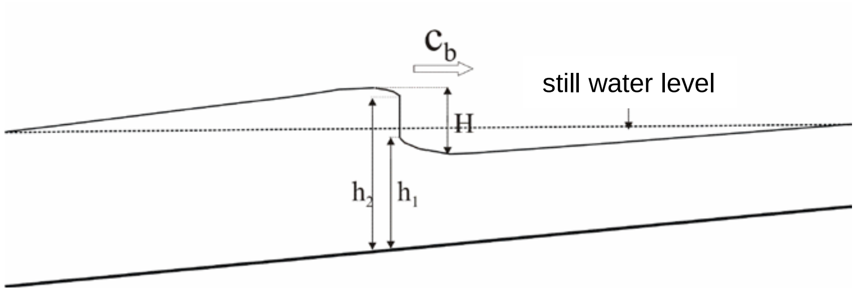


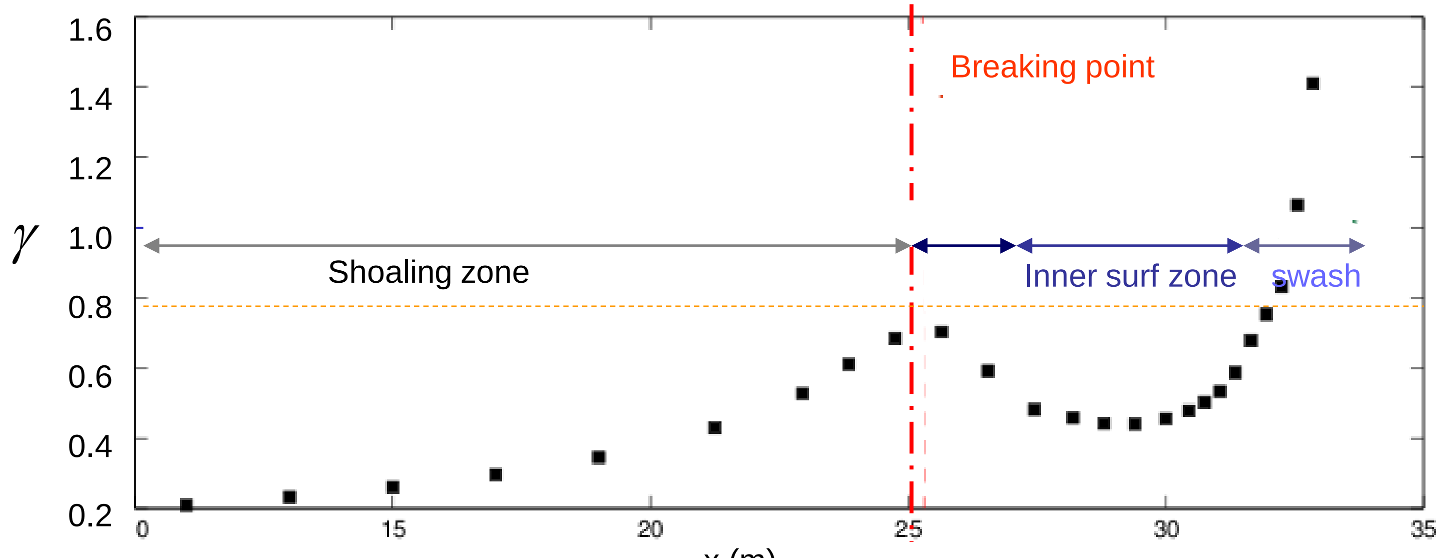
Fig. 3. Snapshots of surface elevation at t = 17.4, 18.6 and 19.9 s from models with grid resolutions of dx = 0.025 (solid lines) and 0.050 m (dashed lines).

❑ Tonelli and Petti (2009, 2010, 2011, 2012)

wealky nonlinear Boussinesq equations

$$\gamma = \frac{H}{d} > 0.8$$





bichromatic wave experiments by van Noorloos (2003)

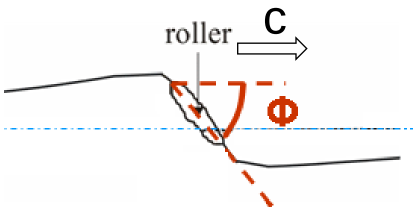
Local treatment of wave breaking

❑ **Tissier et al. (2010, 2012)** → SURF-GN code

- **Initiation of breaking**
  - $\Phi > \Phi_i$

    Φ_i: critical slope

    $\Phi_i = 30°$ (*Cienfuegos et al., 2010*)



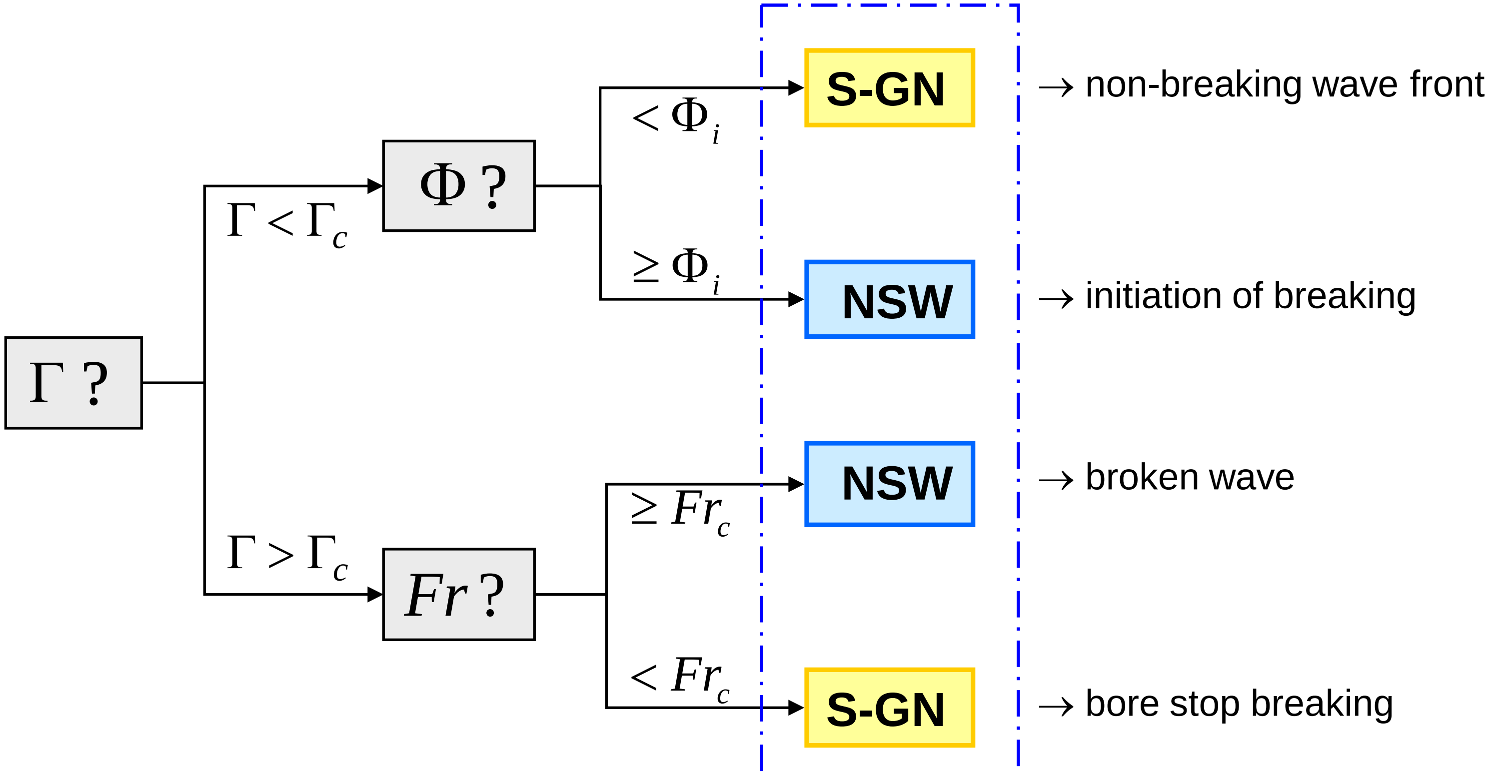
  - $F_r > F_{c1}$

$$Fr = \frac{c - u_1}{\sqrt{gh_1}}$$

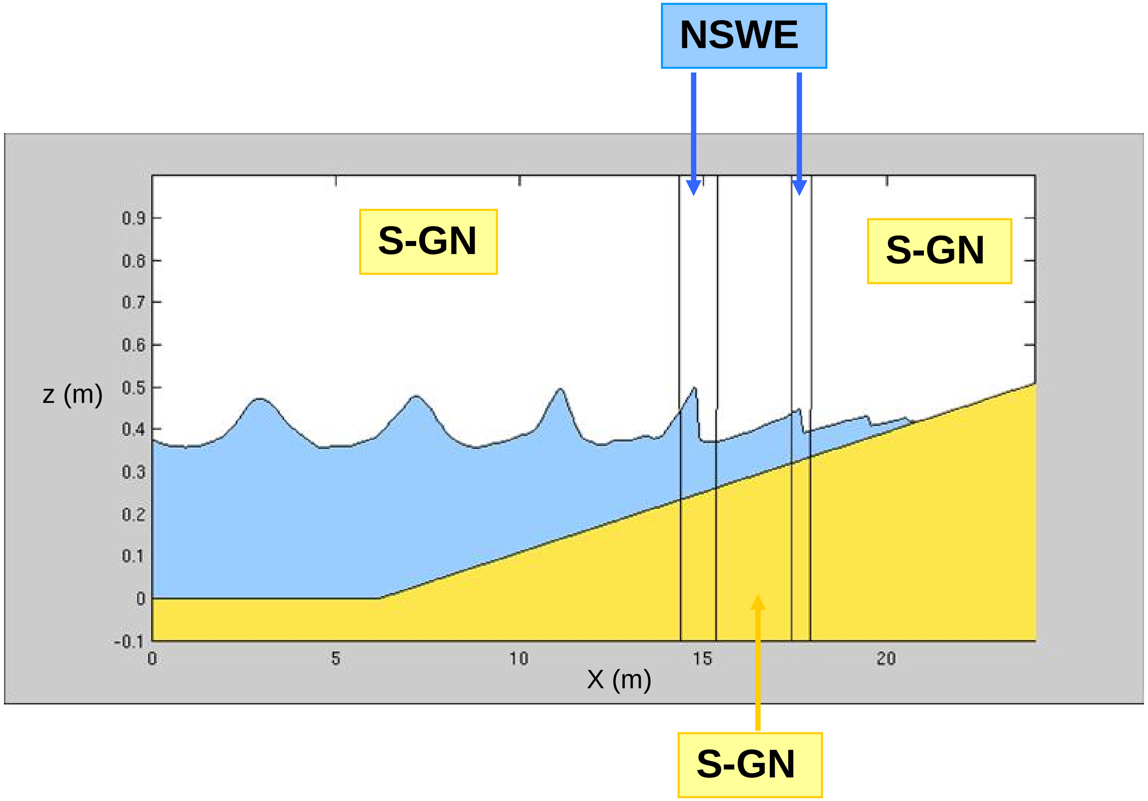
  - $u_c \geq c$

- **Bore stop breaking**
  - $F_r < F_{c2}$

    $F_{c2} = 1.3$ (*Favre 1935, Treske 1994*)

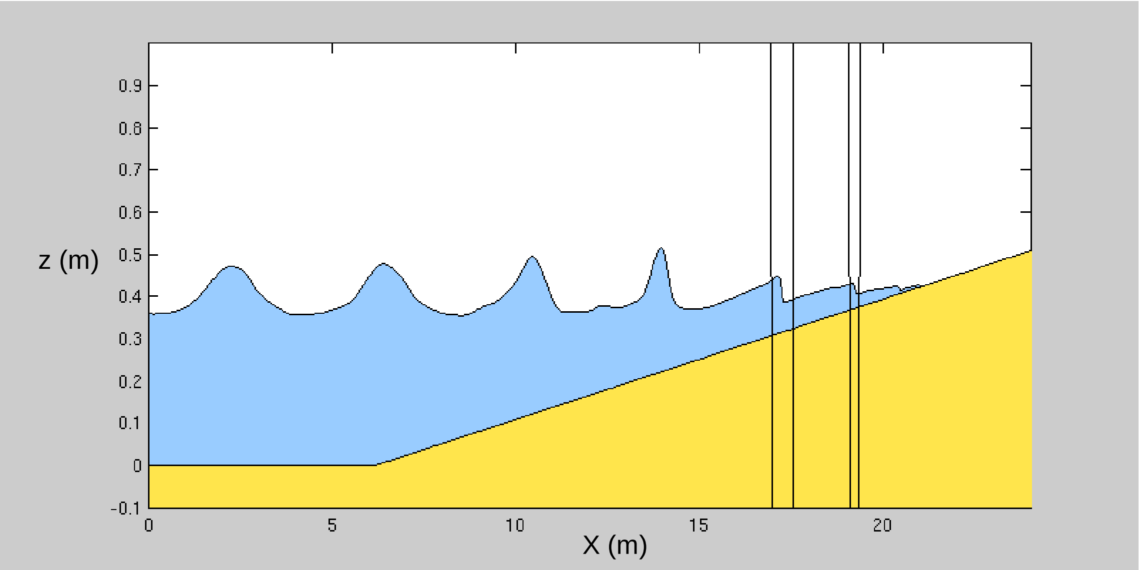

Γ ?
Γ < Γc
Γ > Γc
Φ ?
Fr ?
< Φi
≥ Φi
≥ Frc
< Frc
S-GN
NSW
NSW
S-GN
→ non-breaking wave front
→ initiation of breaking
→ broken wave
→ bore stop breaking

Shoaling and breaking of regular waves over a sloping beach



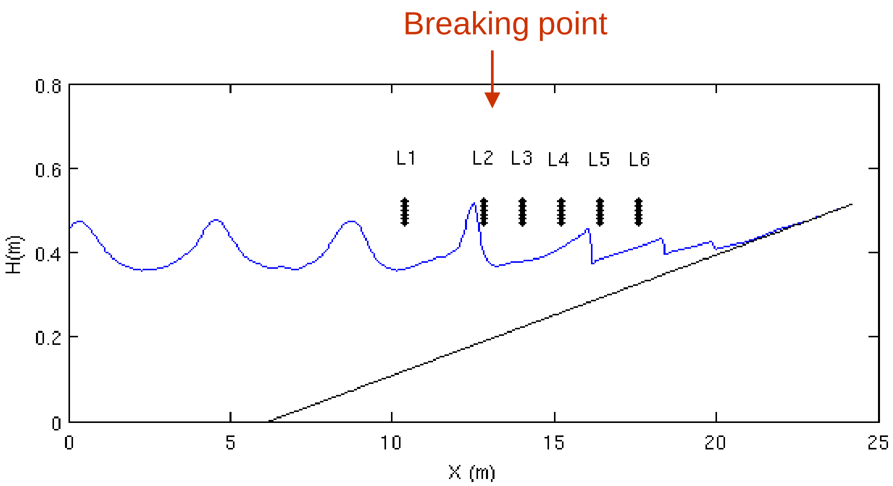
Tissier et al. (2012)

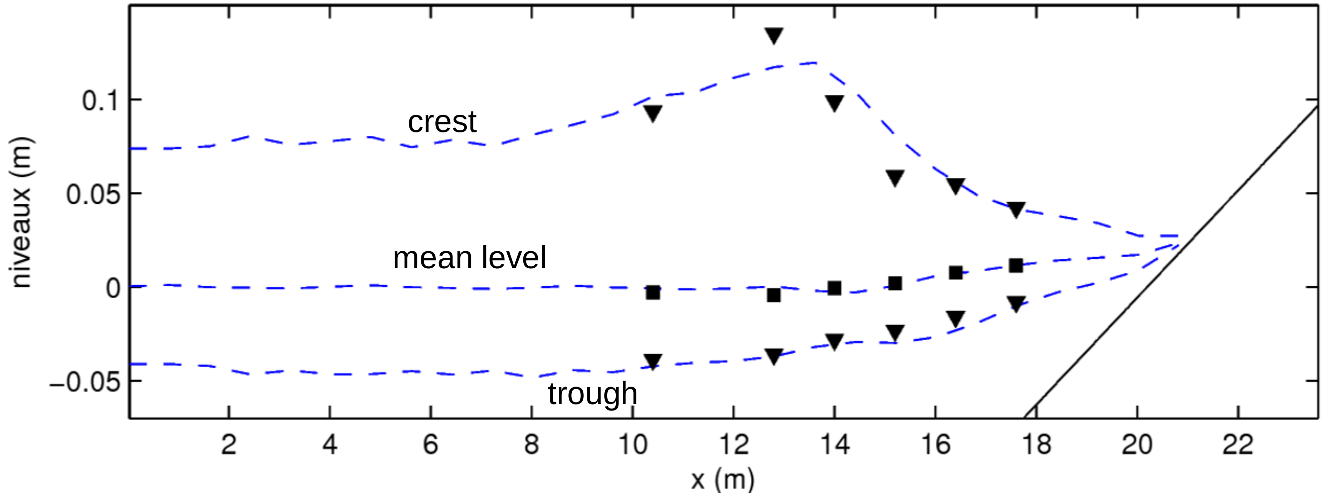
Shoaling and breaking of regular waves over a sloping beach



Shoaling and breaking of regular waves over a sloping beach

Validation with Cox (1995) experiments





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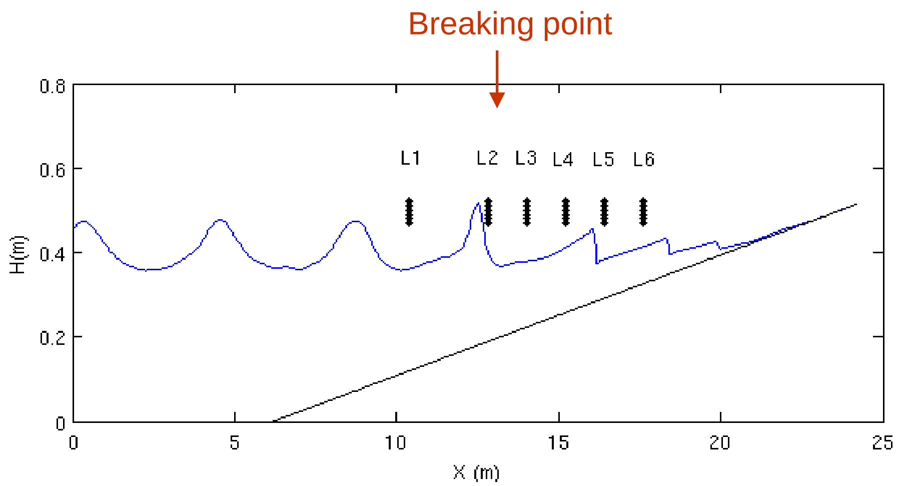
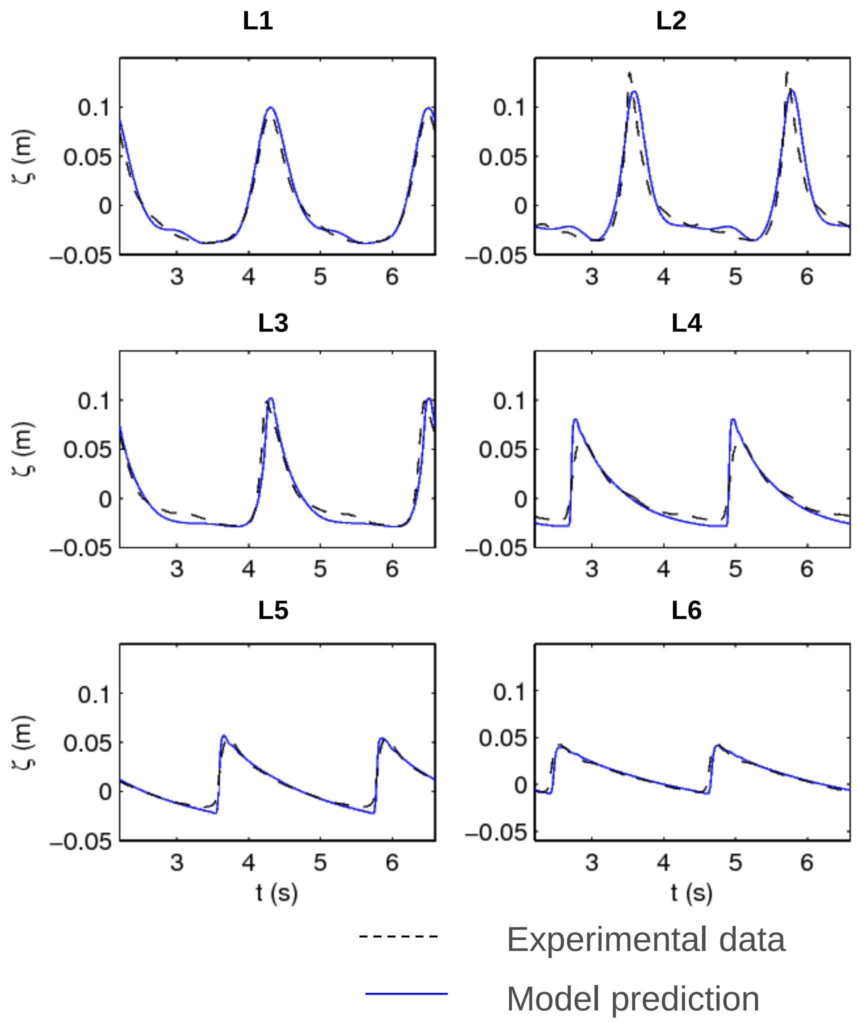
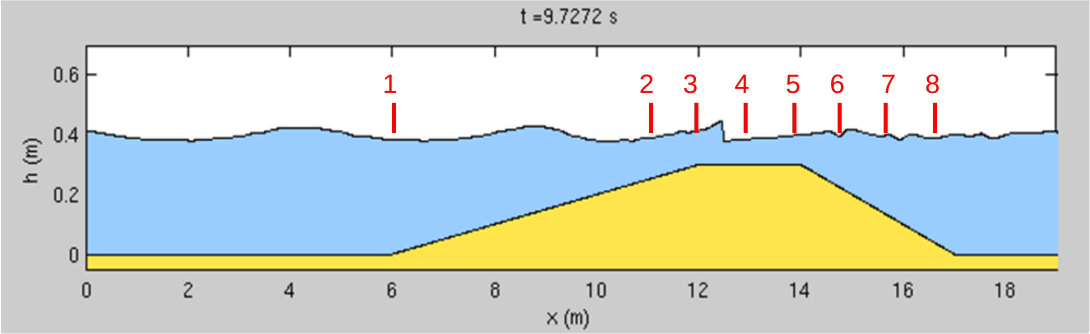




Periodic waves breaking over a bar

Validation with Beji and Battjes (1993) experiments



Tissier et al. (2012)

Periodic waves breaking over a bar

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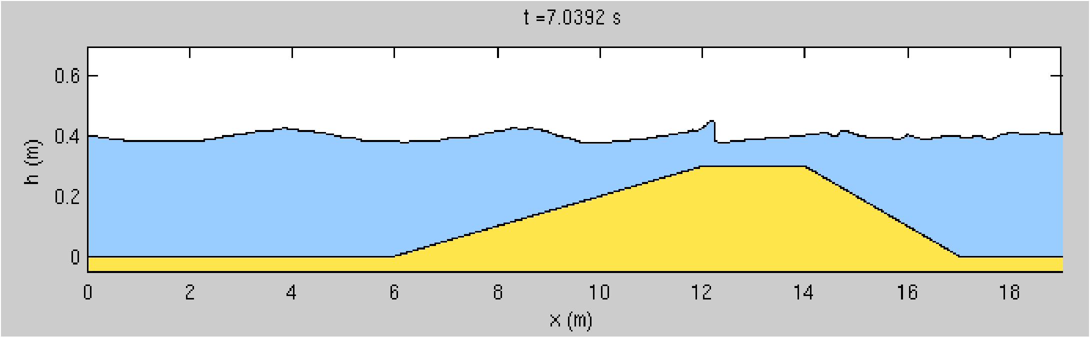
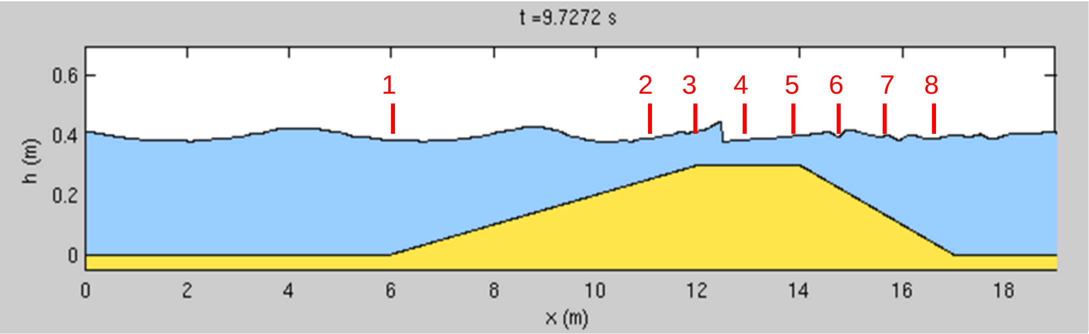
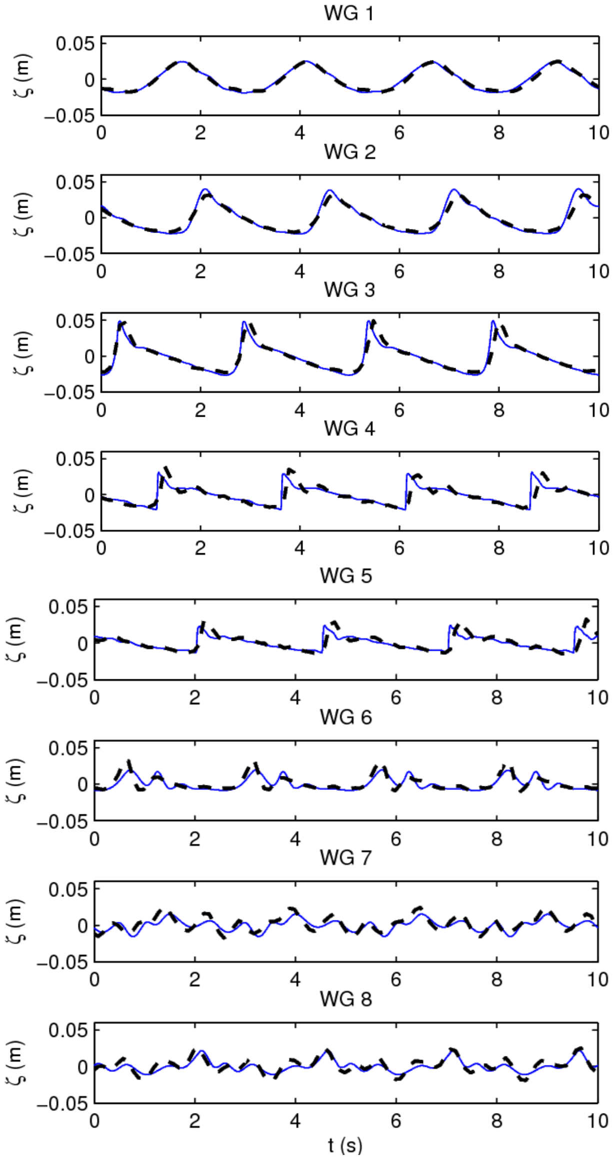


Tissier et al. (2012)

Validation with Beji and Battjes (1993) experiments



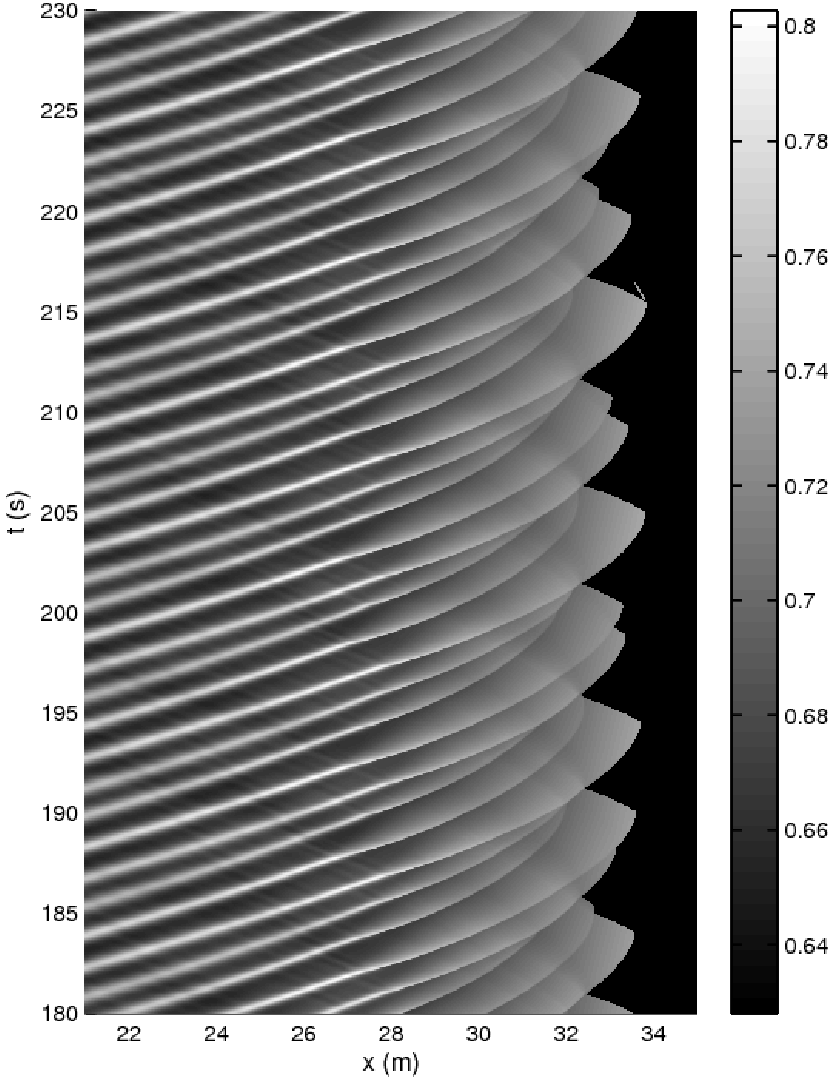


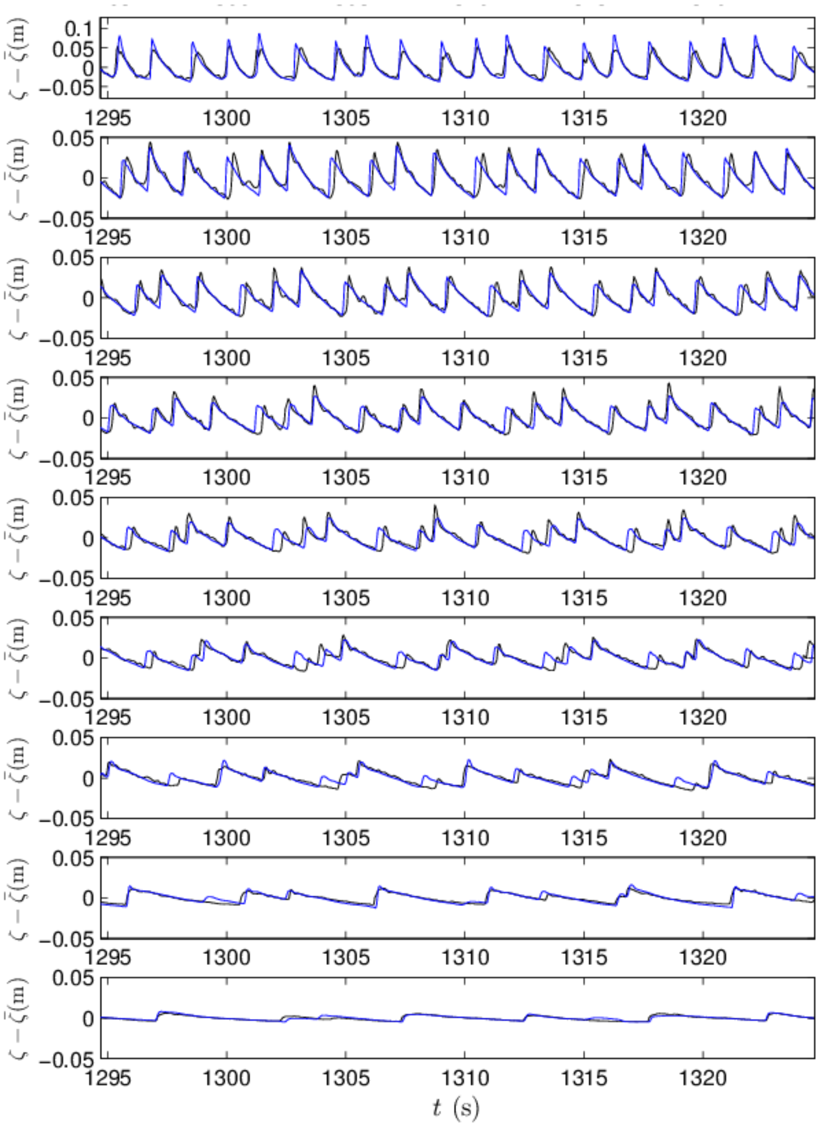
------ Laboratory data

—— Model prediction

Infragravity wave transformation over a low-sloping beach

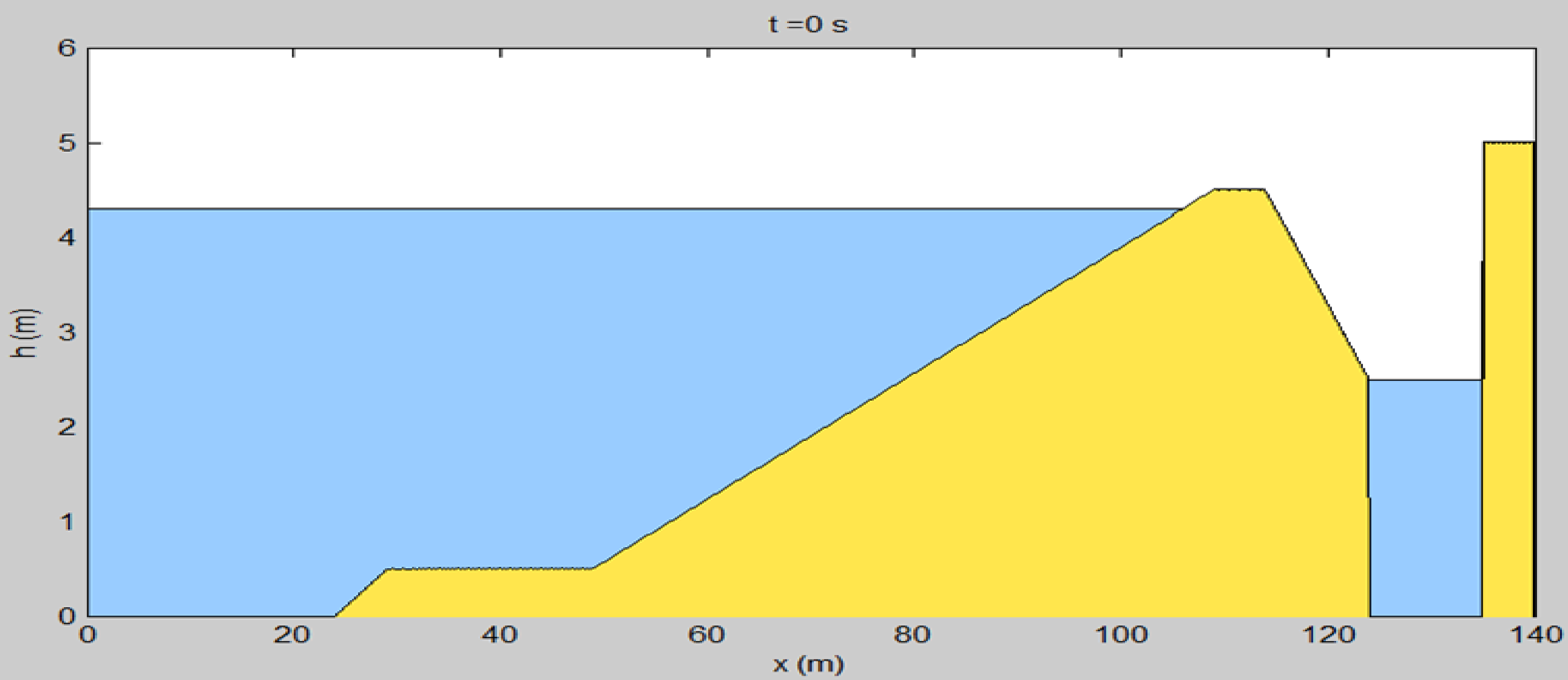
Tissier et al., ICCE 2012 - Friday July 6





Wave overtopping and multiple shorelines

BARDEX II (HYDRALAB project, Delta Flumes, PI: Gerd Masselink)
Barrier Dynamics Experiment : shallow water sediment transport processes in the inner surf, swash and overwash zone.



Wave overtopping and multiple shorelines

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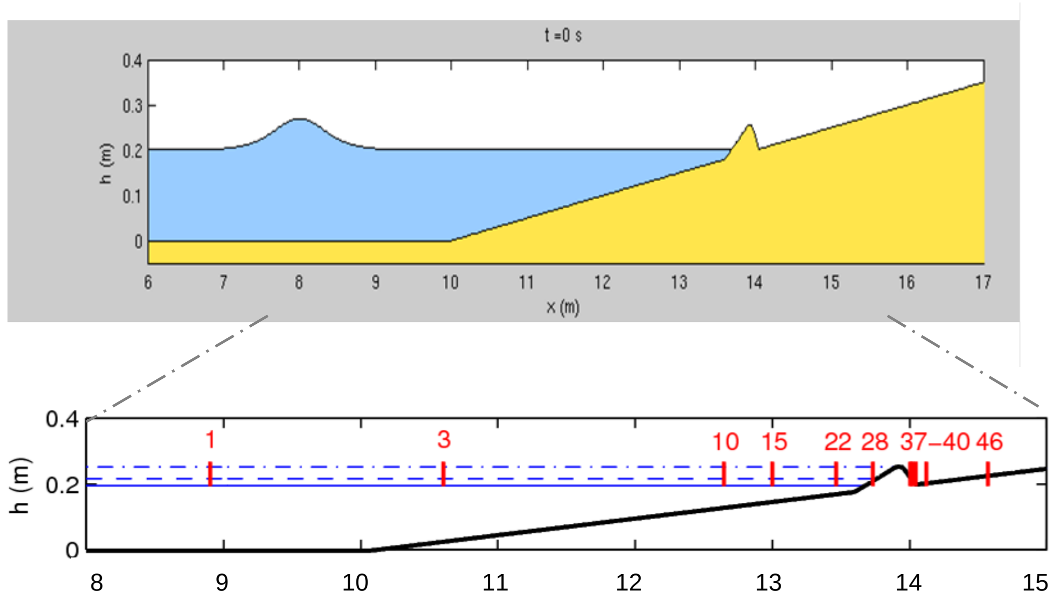
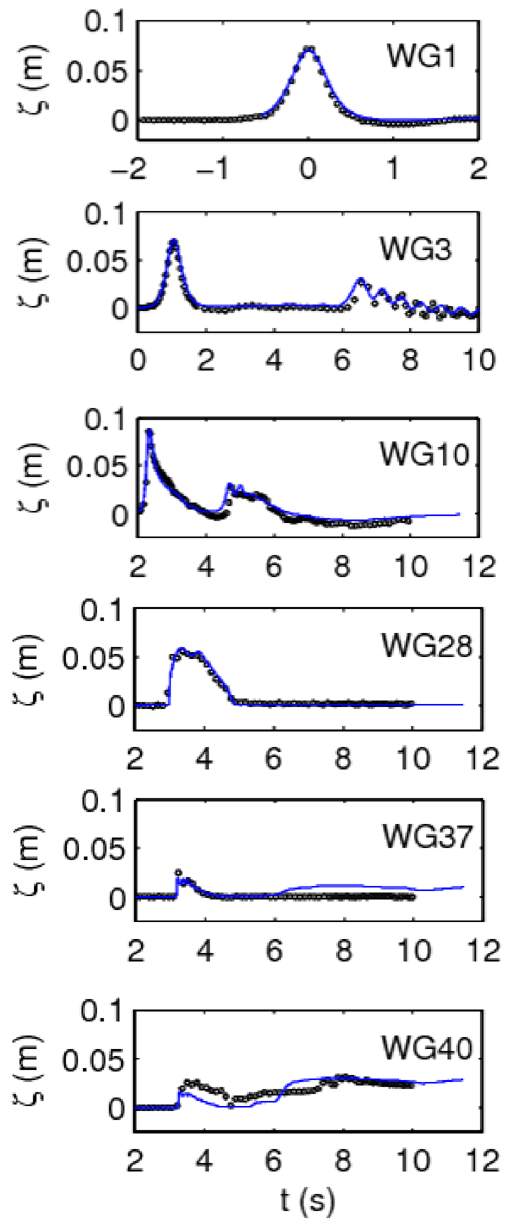
Barrier Dynamics Experiment : shallow water sediment transport processes in the inner surf, swash and overwash zone.

Wave overtopping and multiple shorelines

Solitary waves overtopping a seawall (Hsiao and Lin, 2010)





Wave overtopping and multiple shorelines

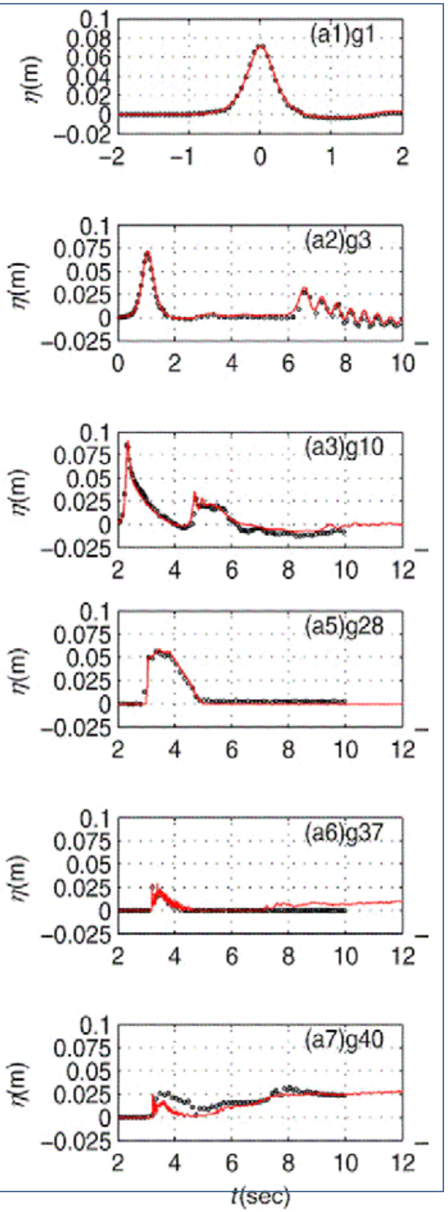
Hsiao et Lin (2010)

COBRAS model

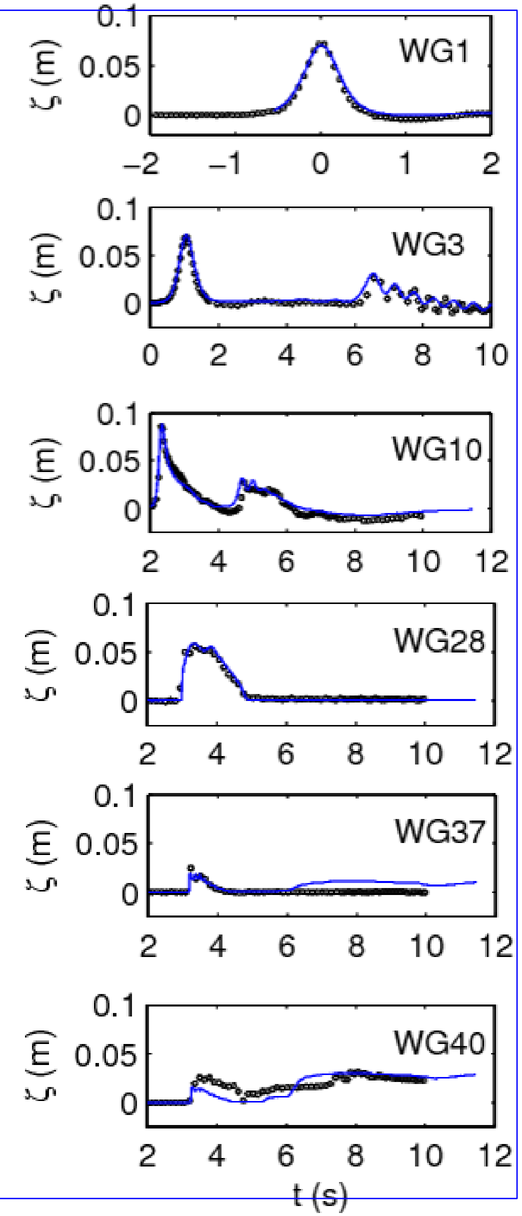
→ see Javier L. Lara IH-2VOF model

2D VOF model

RANS equations K-ε



SURF-GN



FUNWAVE-TVD

Shi et al. (2012)

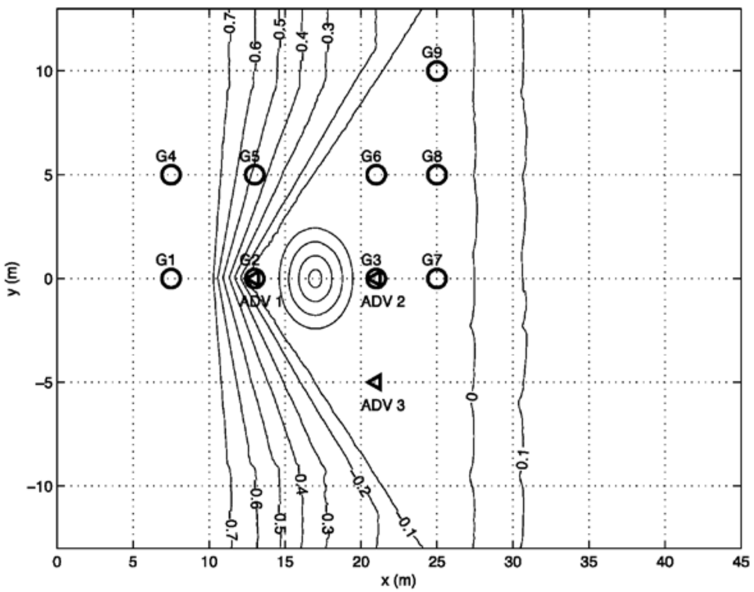


Fig. 8. Bathymetry contours (in meters) and measurement locations used in model simulations for OSU tank bathymetry (Lynett et al., 2010). Circles: pressure gauges, triangles: ADV.

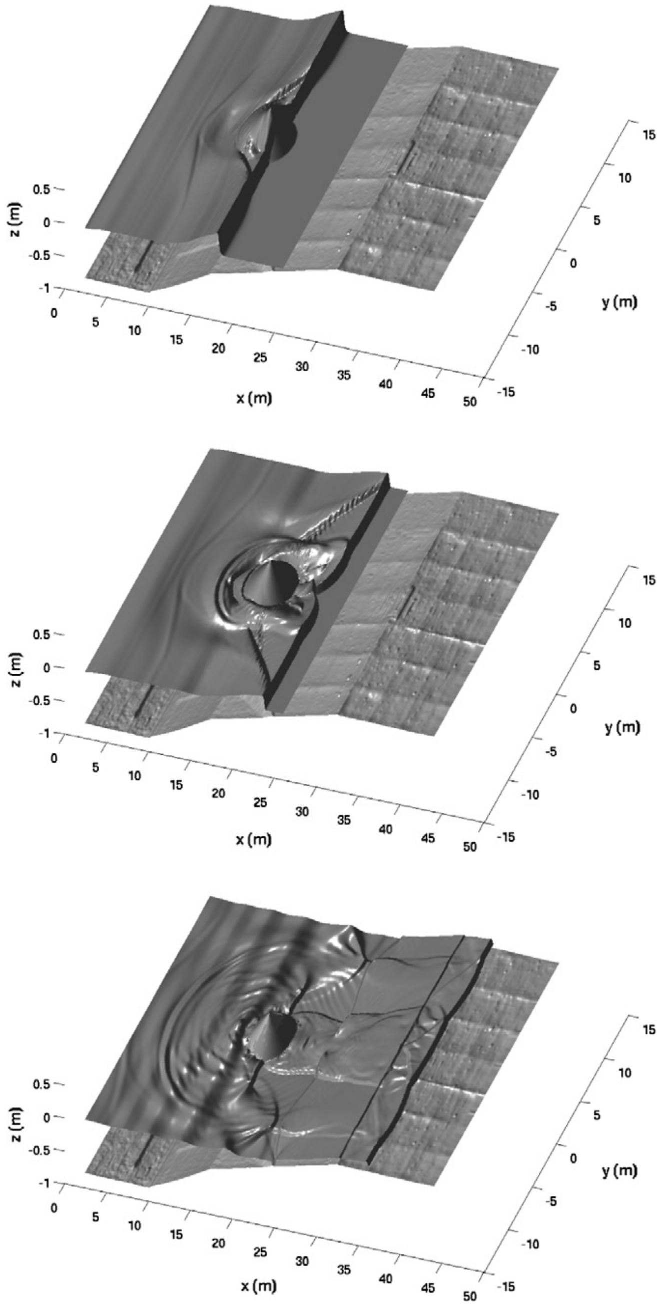


Fig. 9. Modeled water surface at (top) t = 6.4 s, (middle) t = 8.4 s, (bottom) t = 14.4 s.

Landslide tsunami applications

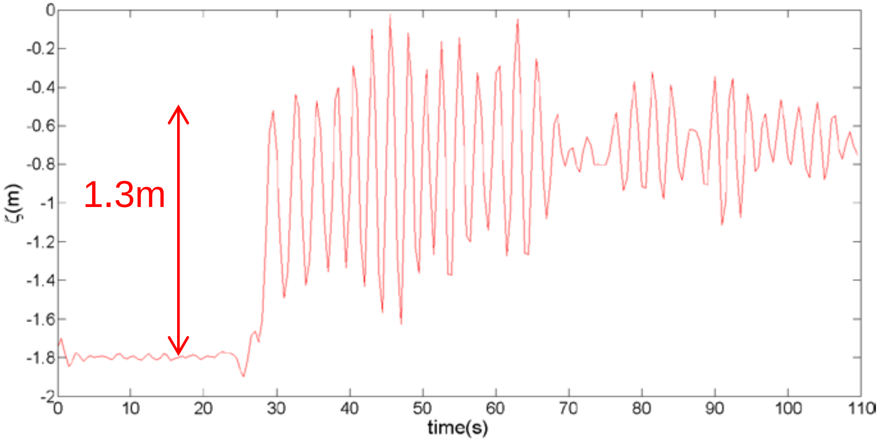
→ *Abadie et al. (2012)*

Conclusion

- Serre-Green Naghdi equations represent the basic fully nonlinear weakly dispersive Boussinesq equations
- clarification about Boussinesq-type equations ☛ to promote interaction between different scientific communities working with these models: oceanography, hydraulics, physics and mathematics
- a new approach for solving S-GN equations
  - new mathematical formulation: conservative variables with only second order derivatives
  - hybrid FV/FD scheme
  - breaking waves and bores described by the NSW shock-wave theory
  - shock-capturing scheme is robust and no filtering is needed
  - benefits from the regular progress of shock-capturing FV methods for NSWE
  - limits the use of ad hoc parametrizations and tuning parameters
- good results for nonlinear wave transformation, wave breaking, swash motions and overtopping, even with multiple shorelines

Perspective

- more validations of the 2DH S-GN approach
  - wave breaking, swash motions and overtopping over complex 3D bathymetries
  - tsunamis or tidal bores propagating up estuaries



Perspective

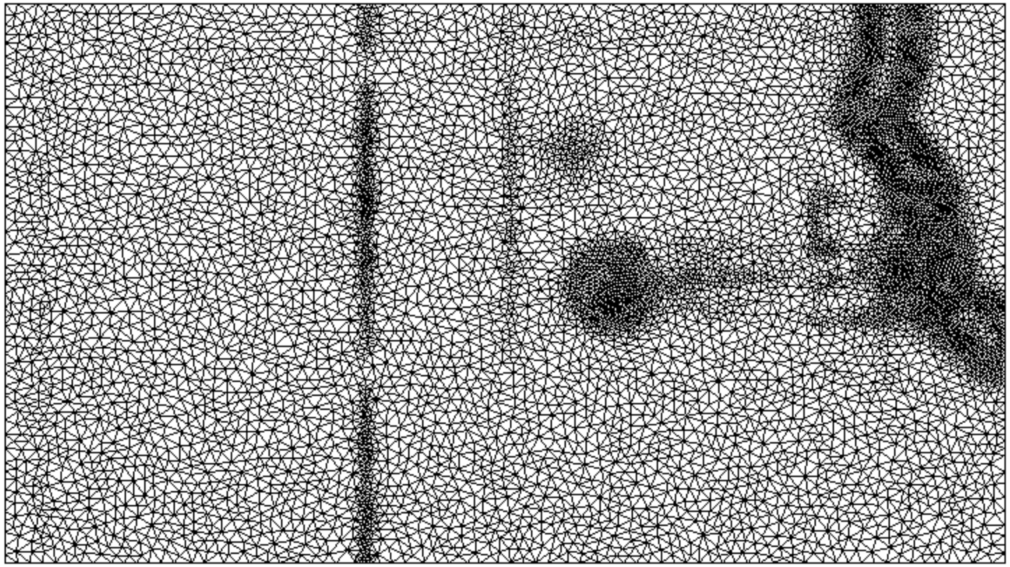
❑ development of Finite Volume or Finite Element methods on unstructured grid

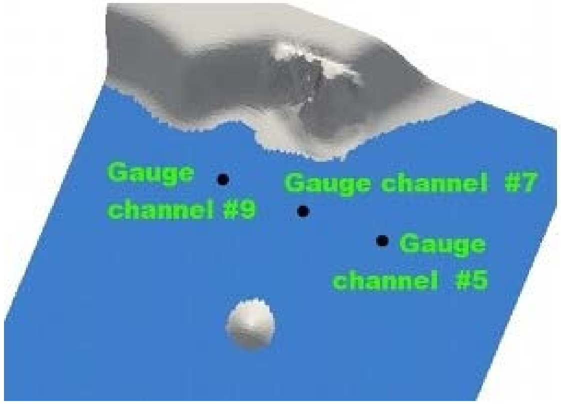
☛ ex.: Mario Ricchiuto (INRIA, Bordeaux) and Fabien Marche

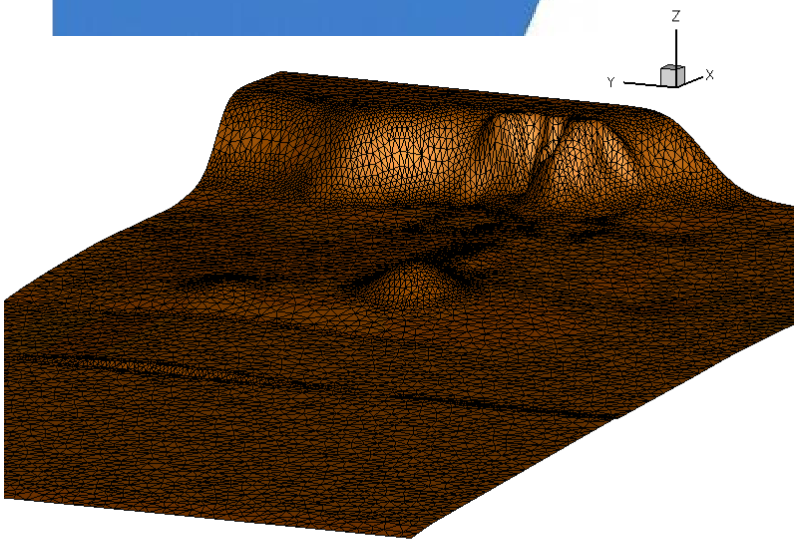
S-GN equations solving with a Discontinuous Galerkin method

Benchmark Problem : Tsunami runup

over a complex 3D beach







Perspective

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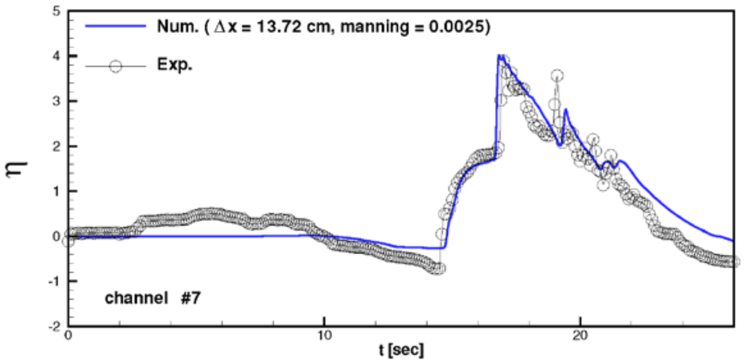
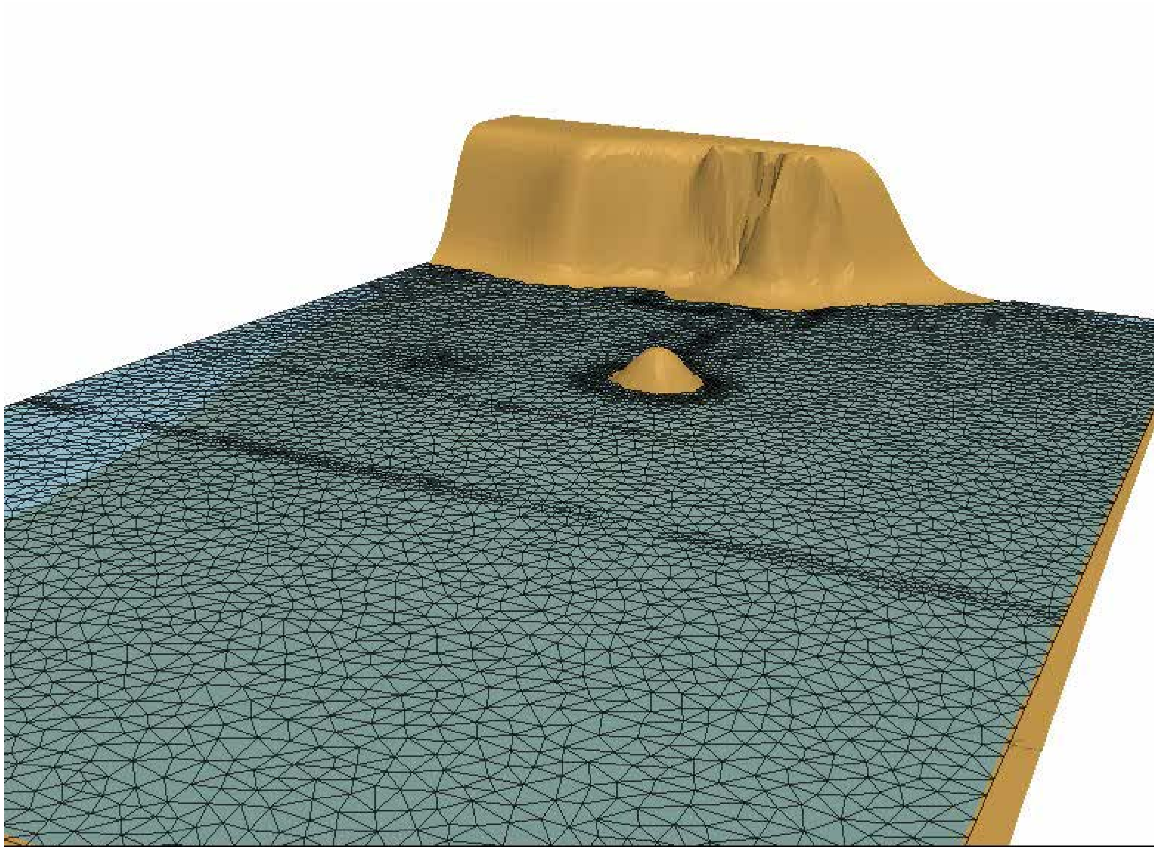




Thank you for your attention

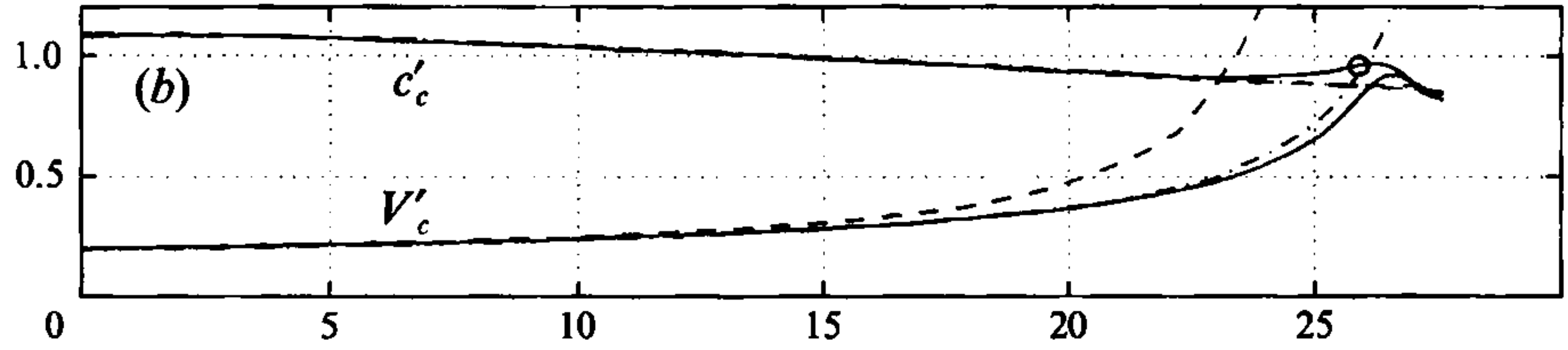
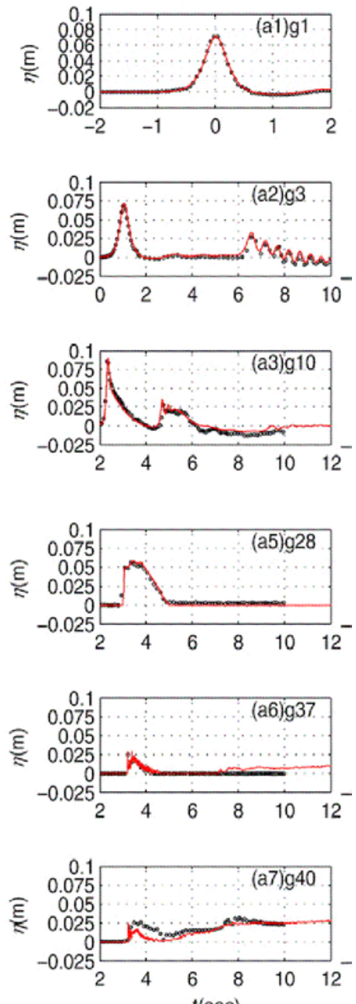


FIGURE 7. Comparison between FNPF (——), BM (- - - - -), and FNBM (— - —) of wave crest celerity c'_c and particle velocity at the crest (components (u'_c, w'_c); value V'_c), for the same solitary waves and slopes as in figure 4. Symbols (○) are defined as in figure 5.

Conclusion

- good results for nonlinear wave transformation, wave breaking, swash motions and overtopping, even with multiple shorelines

VOF model
RANS equations



S-GN equations